Smooth Cubic 3folds Not Rational

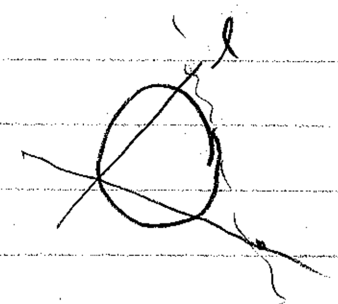
A 3fold V is unirational if there is a finite to one map $\mathbb{P}^3 \dashrightarrow V$.

$$V \subseteq \mathbb{P}^4 \qquad W = \{(p, L) \mid L \in T_p V\}$$
$$\cup \qquad\qquad\qquad\qquad \subseteq \ell \times G(1,4)$$
$$\ell$$

$$W \overset{2:1}{\dashrightarrow} V$$

$(p, \ell) \longmapsto$ residual contact point of $\ell \cap V$

$\overline{x, \ell} \cap V$



V w/ single double point x_0
$\Rightarrow$ N is rational (proj from x_0
$V \overset{\sim}{\dashrightarrow} \mathbb{P}^3$)

C genus g
dim $H^0(K_C) = g$

$p_0 \in C \quad \forall \; p \in C$ $\quad p_0 \overset{\gamma}{\longrightarrow} p \leadsto$ elt of $H^0(K_C)^*$

Get $C \longrightarrow J(C) = \dfrac{H^0(K_C)^*}{H_1(C, \mathbb{Z})}$ by $\int_\gamma \omega$

Abel-Jacobi Jacobian

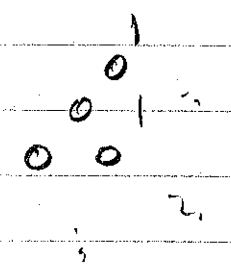
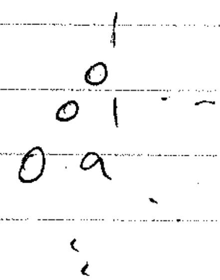
②

$$C^{(g-1)} \longrightarrow J(C) \supseteq \Theta_C$$

<u>Riemann's Sing Thm:</u> Θ_C sing at D
$\Updownarrow$
$\dim |D| > 0$

$$\geq \rho = g-(r+1)(g-d+r)$$
$$\geq g-4$$

$\therefore$ $g \geq 5 \Rightarrow \Theta_C$ has 1-dim family of sing pts

$\mathbb{P}^3$  V 

$0 \to T_V \subseteq T_{\mathbb{P}}|_V \to N_V \to 0$

$\mathcal{O}(3)$

$(1+H)^5 \quad (1+3H)$

$$c(T_V) = (1+H)^5(1-3H+9H^2-27H^3)$$

$$c_3(T_V) = -2H_V^3 = -6 \qquad \therefore\ a = 5$$

$$\begin{tikzcd} & B \arrow[dl] \arrow[dr] & \\ \mathbb{P}^3 \arrow[rr, dashed, "\sim"] & & V \end{tikzcd}$$

$H^*(B)$ gen by $H^*(\mathbb{P}^3) + H^*(E)$

gen by H^*C and ξ

(3)

$\longrightarrow$ " $*$ "

blowup $\quad * \; *^{+1}$

of gns $\quad * \; *_{+g}$

g curve

V smooth proj 3-fold

$\gamma \in H_3(V) \qquad H_3(V) \hookrightarrow (H^{3,0}(V) \oplus H^{2,1}(V))^*$

V sm cubic 3-fold $\Rightarrow$ $(J(V), \Theta_V)$ PPAV $\qquad \Theta_V$

$$J(V) = \frac{(H^{3,0}(V) \oplus H^{2,1}(V))^*}{H_3(V)}$$

(T, Θ_T) PPAV

irred if for any $T' \longrightarrow T \quad T' = 0$ or isom.

T has a unique decomp into irreducible PPAV. Θ_T irred $\Rightarrow$ T is too.

$$J(V) \underset{\cong}{\hookrightarrow} J(C_1) \oplus \cdots \oplus J(C_r)$$

irreducibility

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V cubic 3fold

$$\{(\ell, V) : \ell \subseteq V\} \subseteq G(1,4) \times \mathbb{P}^{34}$$

$$\begin{array}{ccc} & \swarrow \quad \searrow & \\ G(1,4) & & \mathbb{P}^{34} \end{array}$$

S_V = lines on V

exp dim $S_V = 2$

$$T_\ell S = H^0(N_{\ell \subseteq V})$$

$$0 \to N_{\ell/V} \to N_{\ell/\mathbb{P}^4} \to N_{V/\mathbb{P}^4} \to 0 \,\Big|_\ell$$

$$\mathcal{O}(a) \oplus \mathcal{O}(b) \qquad \mathcal{O}(1)^3 \qquad \mathcal{O}(3)$$

$a + b = 0$

$a, b \leq 1 \implies H^0(N_{\ell/V}) = 2$

$\mathcal{O} \oplus \mathcal{O} \qquad \mathcal{O}(-1) \oplus \mathcal{O}(1)$

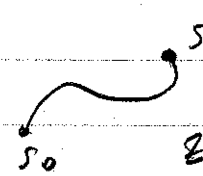
S irred, smooth, 2-dim

$S \times S$ 4 dim irred smooth

$$\alpha : S \longrightarrow J(V)$$

$s_0 \in S$

$$s \longmapsto \int_{s_0}^{s} \omega$$



$$\Sigma \subseteq S \times V$$

$$\begin{array}{ccc} & \swarrow \quad \searrow & \\ S & & V \end{array}$$

$$\Phi : S \times S \xrightarrow[\text{gen } 6:1]{} J(V) \qquad \operatorname{im} \Phi = \Theta, \ 4 \text{dim}$$

$$(s, t) \longmapsto \alpha(s) - \alpha(t) \qquad \text{sing double pt at}$$