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Seçkin

Lefschetz hyperplane theorem (LHT).

X = smooth proj. var, dim $X = n$.

$D \geq 0$ ample divisor on X.

Thm (LHT, version 1).

$$H^k(X, \mathbb{Z}) \longrightarrow H^k(D, \mathbb{Z})$$

this is isom for $k < n-1$

injective for $k = n-1$.

Rmk. A C^∞-mfd of real dim $2n$ has the homotopy type of a CW complex of real dim $\leq 2n$. The following says that complex affine manifolds have only half as much topology:

Thm (Andreotti - Frankel)

$Y \subseteq \mathbb{C}^N$ closed n-dim'l submanifold. Then Y has the htpy type of a CW complex of real dim $\leq n$

Hence $H^k(Y, \mathbb{Z}) = 0$ & $H_k(Y, \mathbb{Z}) = 0$

for $k > n$.

c.f. Milnor, "Morse theory".

<u>Pf of LHT version 1</u> :

Since D ample, mD is very ample for some $m \in \mathbb{N}$ and $\exists$ embedding $X \subseteq \mathbb{P}^N$ and a hyperplane H s.t $mD = X \cap H$.

Then $Y \triangleq X \setminus D = X \setminus mD$ is a smooth complex affine variety of dim n.

By <u>thm A-F</u>, $H_k(Y, \mathbb{Z}) = 0$ for $k > n$.

Using Alexander-Lefschetz duality :

$$H_k(Y; \mathbb{Z}) \simeq H^{2n-k}(X, D; \mathbb{Z})$$

$$\Rightarrow \quad H^k(X, D; \mathbb{Z}) = 0 \quad \text{for } k < n.$$

Now the result follows from

$$\cdots \to H^k(X, D; \mathbb{Z}) \longrightarrow H^k(X; \mathbb{Z}) \to$$
$$H^k(D; \mathbb{Z}) \longrightarrow H^{k+1}(X, D; \mathbb{Z}) \longrightarrow \cdots$$

$\square$

<u>Thm</u> (LHT2). $H_k(D;\mathbb{Z}) \longrightarrow H_k(X;\mathbb{Z})$

is $\cong$ for $k < n-1$

surjective for $k = n-1$. $\square$

<u>Cor 1</u> $X \subseteq \mathbb{P}^{n+c}$ smooth complete intersection. dim $X = n$.

- $H_k(X,\mathbb{Z}) \longrightarrow H_k(\mathbb{P}^{n+c},\mathbb{Z})$

  is an isom for $k<n$.

  surjective for $k=n$.

- $H^k(\mathbb{P}^{n+c},\mathbb{Z}) \longrightarrow H^k(X,\mathbb{Z})$

  is an isom for $k<n$

  injective for $k=n$.

<u>Pf</u>. Do induction on c.

When $c=1$. this is LHT.

Need to show

$$X' = \mathbb{V}(f_1, \dots, f_{c-1})$$

is smooth, where $X = \mathbb{V}(f_1,\dots,f_c) \subseteq \mathbb{P}^{n+c}$.

Can replace $f_1,\dots,f_{c-1}$ by general forms

$f_1', \dots, f_{c-1}'$

of the same degrees resp.

By Bertini, $\exists\, X' = \mathbb{V}(f_1', \dots, f_{c-1}')$ is smooth

Since X is a complete intersection in X', any singular pt of X' lying on X is a sing. pt of X.

So can take X' to be smooth.

Proceed by induction. $\square$

Cor 2. $X \subseteq \mathbb{P}^N$ sm. complete int. of codim n.

$Z \subseteq X$ subvar. of codim $k < \frac{n}{2}$.

Then $\deg X \mid \deg Z$

Pf. By cor 1, $[Z] \in H^{2k}(X, \mathbb{Z})$

$[Z] = (\alpha \cdot \zeta^k)\big|_X$ where $\alpha \in \mathbb{Z}$,

ζ is the class of $X \cap H$, $H \subset \mathbb{P}^N$ hyperplane.

$\Longrightarrow \deg Z = \deg([Z] \cdot \zeta^{n-k}) = \alpha \cdot \deg X$. $\square$

Cor 3. $X \subseteq \mathbb{P}^N$ sm. complete int. of dim $n \geq 3$.

$Z \subseteq X$. subvar. of codim 1.

Then $Z = X \cap H$ for a hypersurface $H \subseteq \mathbb{P}^N$.

Pf. Since $n \geq 3$, by LHT.

$$H^2(\mathbb{P}^N, \mathbb{Z}) \twoheadrightarrow H^2(X, \mathbb{Z})$$

Every line bundle on X is determined by its first Chern class in $H^2(X, \mathbb{Z})$. So every l.b is of the form $\mathcal{O}_X(m)$ for some m.

Suppose $X \subseteq \mathbb{P}^{n+1}$ is a hypersurface $\deg X = d$.

By sequence:

$$0 \to \mathcal{O}_{\mathbb{P}}(m-d) \to \mathcal{O}_{\mathbb{P}}(m) \to \mathcal{O}_X(m) \to 0$$

and the fact $H^1(\mathcal{O}_{\mathbb{P}}(m-d)) = 0$.

$$\Rightarrow \quad H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(m)) \twoheadrightarrow H^0(X, \mathcal{O}_X(m))$$

$\Rightarrow \quad \mathcal{O}_X(Z) = \mathcal{O}_X(m)$ for some m.

and a global section vanishing on Z is the restriction to X of a homog. poly. of degree m.

$\square$

Thm (LHT 3).

$$\pi_k(D) \longrightarrow \pi_k(X) \quad \text{is} \begin{cases} \text{Isom} & \text{for } k < n-1 \\ \text{surj.} & \text{for } k = n-1 \end{cases}$$

Thm (LHT 4). If $\dim X \geq 4$.

$$\mathrm{Pic}(X) \cong \mathrm{Pic}(D)$$

Rmk (Larsen, 1973).

$X \subseteq \mathbb{P}^{n+c}$ sm. subvar. codim $X = c$. Then

$$H^k(\mathbb{P}^{n+c}, \mathbb{Z}) \longrightarrow H^k(X, \mathbb{Z})$$

Isom for $k < n-c$

injective for $k = n-c+1$.