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# Weighted Projective Spaces (wps)

**Def** wps is quotient of $\mathbb{A}^n \setminus \{0\} / k^\times = \mathbb{G}_m$

$$\lambda \curvearrowright (x_1, \dots, x_n) = (\lambda^{a_1} x_1, \dots, \lambda^{a_n} x_n)$$

$(a_1, \dots, a_n)$ positive integers

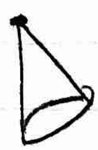
E.g. (1) $\mathbb{P}(1, \dots, 1) = \mathbb{P}^{n-1}$

(2) $\mathbb{P}(1,1,2)$

$$\mathbb{A}^3 \setminus \{0\} \xrightarrow{f} \mathbb{P}^3 \quad z_0\ z_1\ z_2\ z_3$$

$$(x,y,z) \longmapsto (x^2 : xy : y^2 : z)$$

$z_0 z_2 - z_1^2$



$f(x,y,z) = f(x',y',z')$ iff $\exists \lambda$ s.t.

$x = \lambda x'$

$y = \lambda y'$

$z = \lambda^2 z'$

(so not smooth)

**Prop:** $\mathbb{P}(a_0, \dots, a_n)$ smooth $\iff$ it's actually $\mathbb{P}^n$

E.g.: $\mathbb{P}(1,1,r)$

$$\mathbb{A}^3 \setminus \{0\} \longrightarrow \mathbb{P}^{r+1}$$

$$(x_0, \dots, x_2) \longmapsto (x_0^r : x_0^{r-1} x_1 : \dots : x_2)$$

= cone over rational normal curve of degree $r$

②

Eg. $\mathbb{P}(1,\dots,1,r)$ = cone over the veronese

<u>Wps as proj of a graded ring</u>:

$S = k[x_0,\dots,x_n]$ wt $x_i = a_i$

$\mathrm{Proj}\, S = \mathbb{P}(a_0,\dots,a_n)$

$$S^{[d]} = \bigoplus_{d \mid n} S_n \Rightarrow \mathrm{Proj}(S^{[d]}) = \mathrm{Proj}\, S$$

<u>Prop</u>: ① $d \mid a_i$ $\forall i$ $\Rightarrow$ $\mathbb{P}(a_0,\dots,a_n)$

$= \mathbb{P}\left(\frac{a_0}{d},\dots,\frac{a_n}{d}\right)$

② If $\gcd(\text{all } a_i\text{'s}) = 1$, $d = \gcd$ of $(a_0,\dots,\hat{a}_i,\dots,a_n)$ $\Rightarrow$ $\mathbb{P}(a_0,\dots,a_n)$

$= \mathbb{P}\left(\frac{a_0}{d},\dots,\frac{a_{i-1}}{d}, a_i, \frac{a_{i+1}}{d}\dots\right)$

<u>Eg.</u> $\mathbb{P}(15,10,6) \cong \mathbb{P}(3,2,6) \cong \mathbb{P}(1,1,2)$

$\cong \mathbb{P}(1,1,1) = \mathbb{P}^2$

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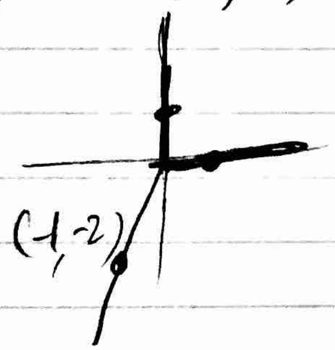
wpr as toric var

$$N = \mathbb{Z}^{n+1} \Big/ \mathbb{Z}(a_0, \ldots, a_n)$$

fan = cones gen by all subsets of $\leq n$ $\vec{e}_i$'s

e.g. $\mathbb{P}(1,1,2)$



Prop: $X = \mathbb{P}(\underline{a})$

$\underline{a} = (a_0, \ldots, a_n)$

w/ no n have common factor

$Cl(X) \cong \mathbb{Z}$

$Pic(X) \cong \mathbb{Z}$ is mult by

$m = \mathrm{lcm}(a_i)$

Pf: $M \longrightarrow \underset{\mathbb{Z}^{n+1}}{Div_{T_N}(X)} \xrightarrow{(b) \mapsto \sum b_i a_i} Cl(X) \longrightarrow 0$

$$m \longmapsto (\langle m, u_0 \rangle, \ldots, \langle m, u_n \rangle)$$

Cor: Every Weil-div is Cartier iff $\mathrm{lcm}(a_i) = 1 \iff$ have $\mathbb{P}^n$.

wps as finite quotient of $\mathbb{P}^n$

$$\bigoplus \mu_{a_i} \curvearrowright \mathbb{P}^n$$

$$[x_0 : \dots : x_n] \longmapsto (\zeta_{a_0} x_0, \dots, \zeta_{a_n} x_n) \text{ etc}$$

$$\mathbb{P}^n / \bigoplus \mu_{a_i} \cong \mathbb{P}(a_0, \dots, a_n)$$

$\Longrightarrow$ cyclic quotient singularities

**Def:** $X \subseteq \mathbb{P}(a)$ well-formed if

$$X \cap \mathbb{P}_{sing}(a) \quad \text{codim} \geq 2 \text{ in } X$$

**Prop:** $X$ a well-formed quasismooth complete $\cap$,

$$\omega_X = i_* \omega_{X^{sm}}$$

and $\omega_X = \mathcal{O}_X(\Sigma d_i - \Sigma a_i)$

where $d_i = \deg f_i$

$$I(X) = \langle f_i \rangle$$

**Prop:** All curves (hypsurf) w/
$d - \sum q_i = 0 \cong$ to

| $q$ | $d$ | eq$^n$ |
|---|---|---|
| $(1,1,1)$ | $3$ | $x^3+y^3+z^3+bxyz$ |
| $(1,1,2)$ | $4$ | $x^4+y^4+z^2+bx^2yz$ |
| $(1,2,3)$ | $6$ | $x^6+y^3+z^2+bx^2y^2$ |