AN INTRODUCTION TO VOLUME FUNCTIONS FOR ALGEBRAIC CYCLES

ABSTRACT. DRAFT VERSION

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1. INTRODUCTION

One of the oldest problems in algebraic geometry is the Riemann-Roch problem: given a divisor L on a smooth projective variety X, what is the dimension of $H^0(X, L)$? Even better, one would like to calculate the entire section ring $\oplus_{m \in \mathbb{Z}} H^0(X, mL)$. It is often quite difficult to compute this ring precisely; for example, section rings need not be finitely generated.

Even if we cannot completely understand the section ring, we can still extract information by studying its asymptotics – how does $H^0(X, mL)$ behave as $m \to \infty$? This approach is surprisingly fruitful. On the one hand, asymptotic invariants are easier to compute and have nice variational properties. On the other, they retain a surprising amount of geometric information. Such invariants have played a central role in the development of birational geometry over the last forty years.

Perhaps the most important asymptotic invariant for a divisor L is the volume, defined as

$$\mathrm{vol}(L) = \limsup_{m \to \infty} \frac{\dim H^0(X, mL)}{m^n/n!}.$$

In other words, the volume is the section ring analogue of the Hilbert-Samuel multiplicity for graded rings. As we will see shortly, the volume lies at the intersection of many fields of mathematics and has a variety of interesting applications.

This expository paper outlines recent progress in the construction of "volume-type" functions for cycles of higher codimension. The main theme is the relationship between:

- the positivity of cycle classes,
- an asymptotic approach to enumerative geometry, and
- the convex analysis of positivity functions.

The theory of volume for higher codimension cycles has been developed in the recent papers [Leh15], [FL13], [Xia15], [LX15a], [LX15b].

1.1. **History.** The volume function for divisors has its roots in the work of Demailly on Fujita's Conjecture ([Dem93]). The first explicit references to the volume in the algebraic setting occurred in [ELN96] (for applications to Fujita's conjecture) and in [Fuj94] (for applications to the Zariski decomposition problem). The theory was then developed systematically in [DEL00] and [Laz04, Chapter 2.2].

Here are a few areas where the volume function has been of use:

(1) Birational geometry. The applications are too varied to summarize here. The volume is particularly useful for analyzing the geometry of the canonical divisor, with applications to Fujita's conjecture ([Dem93], [AS95]) and boundedness statements ([Tsu06], [HM06], [Tak06], [HMX13]).

(2) Complex geometry. The volume has emerged as a powerful tool for understanding the holomorphic morse inequalities and the positivity of singular hermitian metrics. This viewpoint was initially developed in [Bou02b], [Bou04], [BDPP13].

(3) Convex geometry. For toric varieties, the volume is an important link between the geometry of divisors and the theory of convex bodies. In particular, many geometric inequalities have analogues in algebraic geometry which use the volume function ([Tei82], [Kho89]). More recently, this dictionary has been extended to arbitrary varieties using the Okounkov body construction ([LM09], [KK12]).

(4) Enumerative geometry. The volume can be interpreted via enumerative geometry by counting point incidences on divisors (see [DEL00] for a related statement). For higher codimension cycles this perspective raises many new and interesting questions in enumerative geometry. From this viewpoint, the volume is also related by analogy to many recent advances in enumerative problems over finite fields, e.g. versions of the Kakeya conjecture.

(5) K-stability. [Fuj15a], [Fuj15b], [FO16] have shown how the volume function can be used to analyze the existence of Kähler-Einstein metrics on Fano varieties.

(6) Number theory. The arithmetic analogue of the volume function is an important construction in Arakelov theory (see for example [Yua08], [Yua09]). The volume also has appeared in connection with Diophantine problems as in [MR15].

Two excellent introductions to the volume function are [Laz04, Chapter 2.2] and [ELM$^+$05]; we will refer to the former for most of the background results.

2. Notation

Throughout we will work over $\mathbb{C}$ (although many of the results hold for arbitrary algebraically closed fields). A variety is always reduced and irreducible.

2.1. **Numerical spaces.** A k-cycle on X is a finite formal sum $\sum a_i Z_i$ where each $a_i \in \mathbb{Z}$ and each Z_i is a k-dimensional subvariety of X. When X is smooth of dimension n, [Ful84] describes an intersection pairing between k-cycles and $(n-k)$-cycles. Two k-cycles Z_1, Z_2 are numerically equivalent if they have the same intersection number against every cycle of complementary dimension.

We let $N_k(X)_\mathbb{Z}$ denote the free abelian group of k-cycles up to numerical equivalence and denote its tesnor product with $\mathbb{Q}$ by $N_k(X)_\mathbb{Q}$. We usually work with the kth numerical group, which is the finite dimensional vector space

$$N_k(X) := N_k(X)_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{R}.$$

Note that $N_k(X)_\mathbb{Z}$ defines a lattice in $N_k(X)$.

Remark 2.1. When X is singular, there is no longer a natural intersection pairing for cycles, and we define numerical equivalence by intersecting against (homogeneous polynomials in) chern classes of vector bundles instead.

The pseudo-effective cone $\overline{\mathrm{Eff}}_k(X)$ is the closure of the cone in $N_k(X)$ generated by the classes of irreducible k-dimensional subvarieties. [FL15] verifies that it is a pointed convex closed cone. Classes in the interior of the pseudo-effective cone are known as big classes; we denote the set of such classes by $\mathrm{Big}_k(X)$. Given classes $\alpha, \beta \in \overline{\mathrm{Eff}}_k(X)$, we use the notation $\alpha \preceq \beta$ to denote the condition $\beta - \alpha \in \overline{\mathrm{Eff}}_k(X)$.

The movable cone $\mathrm{Mov}_k(X)$ is the closure of the cone in $N_k(X)$ generated by the classes of irreducible k-dimensional subvarieties which deform to cover X. This well-known construction for divisors and curves seems to play in important role in intermediate codimension as well. Its basic properties are studied in [FL13].

2.2. **Divisors.** We let $N^1(X)$ denote the Neron-Severi space of numerical classes of Cartier divisors. We let $\overline{\mathrm{Eff}}^1(X)$, $\mathrm{Big}^1(X)$, $\mathrm{Nef}^1(X)$ and $\mathrm{Amp}^1(X)$ denote respectively the pseudo-effective cone, big cone, nef cone, and ample cone of divisor classes.

The positive product of [Bou02a], [BDPP13] is a useful tool for analyzing the positivity of divisors. We briefly recall the algebraic construction of [BFJ09]. Suppose L is a big $\mathbb{R}$-Cartier divisor class on X. For any birational map $\phi : Y \to X$ and any ample $\mathbb{R}$-divisor class A on Y satisfying $\phi^* L \succeq A$, consider the numerical k-cycle class $\phi_* A^{n-k}$ on X. [BFJ09, Section 2] shows that as we vary over all (ϕ, A), the resulting classes admit a unique supremum under the relation $\preceq$. We denote this class by $\langle L^{n-k} \rangle \in N_k(X)$.

2.3. **Convex analysis.** The Legendre-Fenchel transform describes the strict convexity of a function. It associates to a convex real-valued function f on $\mathbb{R}$ a function f^* on $\mathbb{R}^\vee$; its key property is that the strict convexity of f is related to the differentiability of f^* and vice versa ([Roc70]).

We will work with an analogous construction in a different setting. Suppose that C is a cone in a finite dimensional vector space and that $f : C \to \mathbb{R}$ is a non-negative real function which is homogeneous of weight $s > 1$. The function f is said to be s-concave on the interior of C if for every $v, w \in C^\circ$

$$f(v + w)^{1/s} \geq f(v)^{1/s} + f(w)^{1/s}.$$

This is of course equivalent to the more familiar condition that for every $t \in [0, 1]$ we have $f(tv + (1 - t)w)^{1/s} \geq tf(v)^{1/s} + (1 - t)f(w)^{1/s}$.

The analogue of the Legendre-Fenchel transform for concave homogeneous functions on cones is known as the polar transform. To a positive s-concave homogeneous function on C°, it associates a non-negative $\frac{s}{s-1}$-concave homogeneous function on the interior of the dual cone $C^\vee$.

Definition 2.2. The polar transform of f as above is the function $\mathcal{H}f : C^{\vee\circ} \to \mathbb{R}_{\geq 0}$ defined as

$$\mathcal{H}f(w^*) = \inf_{v \in C^\circ} \left(\frac{w^* \cdot v}{f(v)^{1/s}} \right)^{s/s-1}.$$

The polar transform relates strict log-concavity with differentiability in a similar way as the Legendre-Fenchel transform. However, there can be "extra" failure of strict concavity when f is positive along the boundary of C.

Many of the classical results in convex analysis, such as the Young-Fenchel inequality or the Brunn-Minkowski inequality, have avatars in this setting ([LX15a]).

3. Volume for divisors

Suppose that X is a smooth projective variety of dimension n and L is a Cartier divisor on X. We would like to understand the asymptotic behavior of the section ring $\oplus_m H^0(X, mL)$. We first must establish the expected rate of growth. By restricting to an ample divisor and using the LES of cohomology, one easily shows by induction that

Lemma 3.1. *There is a constant $C = C(X, L)$ such that $\dim H^0(X, mL) < Cm^n$ for every positive integer m.*

Not only is the growth rate bounded above by a degree n polynomial, but every variety carries divisors which achieve this "expected" growth rate. A divisor L whose section ring grows like a polynomial of degree n is known as a big divisor. (We will soon see a divisor is big if and only if its class is big in the sense of Section 2.1, resolving the potential conflict in notation.)

The focus of this section is the asymptotic invariant of L known as the volume:

$$\mathrm{vol}(L) = \limsup_{m \to \infty} \frac{\dim H^0(X, mL)}{m^n/n!}.$$

It is true, but not obvious, that the limsup is in fact a limit ([Laz04, Example 11.4.7]). The rescaling factor $n!$ ensures that the volume of the hyperplane class on projective space is 1. Before discussing this function further, we give several extensive examples.

Example 3.2 (Ample divisors). Suppose that A is an ample divisor. By Serre vanishing and asymptotic Riemann-Roch, we find that for sufficiently large m

$$\dim H^0(X, mA) = \chi(X, \mathcal{O}_X(mA))$$
$$= \frac{A^n}{n!}m^n + O(m^{n-1}).$$

Thus the volume coincides with the top self-intersection A^n.

Note that the volume of an ample divisor only depends on its numerical class, so that the function descends to ample numerical classes in $N^1(X)$. From this perspective, it is natural to extend the volume to all of $\mathrm{Nef}^1(X)$ as the top self-intersection product. We can then consider the analytic properties of this function: it is continuous, homogeneous of degree n, and infinitely differentiable on $\mathrm{Amp}^1(X)$. A key property is the strict n-concavity on the big and nef cone. This is described by the Khovanskii-Teissier inequality, the algebro-geometric incarnation of the Brunn-Minkowski inequality.

Lemma 3.3 ([BFJ09] Theorem D). *Let X be a projective variety of dimension n. Given any two big and nef classes $a, h \in N^1(X)$, we have*

$$\mathrm{vol}(a + h)^{1/n} \geq \mathrm{vol}(a)^{1/n} + \mathrm{vol}(h)^{1/n}$$

with equality if and only if a and h are proportional.

Example 3.4 (Toric varieties). Suppose that X is a smooth toric variety of dimension n defined by a fan Σ in a finite rank free abelian group N. Let r_i denote the rays of the fan and let D_i denote the torus-invariant divisor corresponding to r_i. We let $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ and let $M_{\mathbb{R}}$ denote the dual space.

To any divisor D on X we can associate a polytope whose normal facets are a subset of the rays of Σ. Write $D \sim \sum_i a_i D_i$; then the associated polytope is

$$P_D := \{v \in M_{\mathbb{R}} \mid \langle v, r_i \rangle + a_i \geq 0\}.$$

The volume of D is $n!$ times the volume of the polytope P_D.

This example indicates the tight relationship between volume functions, geometric inequalities, and convex geometry, as pioneered by Teissier ([Tei79], [Tei82]) and Khovanskii ([Kho89]). More recently, this relationship has been extended to arbitrary varieties by the Okounkov body construction ([LM09], [KK12]).

Example 3.5 (Surfaces). Suppose S is a smooth projective surface S. An influential idea of Zariski allows one to replace an effective divisor L by a nef $\mathbb{Q}$-divisor P with the same asymptotic behavior of sections. We describe how this works for big curves:

Definition 3.6 ([Zar62], [Fuj79]). Let L be a big effective curve on S. There exists a unique big and nef $\mathbb{Q}$-divisor P and a unique effective $\mathbb{Q}$-divisor N satisfying:

$$L = P + N \qquad \text{and} \qquad P \cdot N = 0.$$

The decomposition $L = P + N$ is known as the Zariski decomposition of L; experts will readily verify that this definition agrees with the one given by Zariski in terms of the self-intersection matrix of $\mathrm{Supp}(N)$. Note again the numerical nature: the negative part only depends on the numerical class of L.

[Zar62] shows that for any positive integer m, the natural inclusion

$$H^0(S, \lfloor mP \rfloor) \to H^0(S, mL)$$

is an isomorphism. As a consequence, one easily checks that

$$\mathrm{vol}(L) = \mathrm{vol}(P) = P^2.$$

Zariski proved a more precise statement about the asymptotic behavior of sections of P, but we do not need this stronger statement.

It will be useful to recast the Zariski decomposition in terms of the convexity of the volume function. By combining the Zariski decomposition with the strict 2-concavity of the volume on the big and nef cone, we see:

Theorem 3.7. *Let L and D be two big divisors on S. Then*

$$\mathrm{vol}(L + D)^{1/2} \geq \mathrm{vol}(L)^{1/2} + \mathrm{vol}(D)^{1/2}$$

with equality if and only if the numerical classes of P_L and P_D are proportional.

In other words, the Zariski decomposition exactly captures the failure of the strict 2-concavity of the volume function.

3.1. **Basic analytic properties.** We next discuss the basic properties of the volume function.

Proposition 3.8 ([Laz04] Propositions 2.2.35 and 2.2.41). *Let X be a projective variety of dimension n. Then:*
- *The volume is homogeneous of weight n: for any Cartier divisor L and positive integer c we have $\mathrm{vol}(cL) = c^n \mathrm{vol}(L)$.*
- *If L and D are numerically equivalent Cartier divisors, then $\mathrm{vol}(L) = \mathrm{vol}(D)$.*

The second property shows that the volume can naturally be considered as a function on $N_1(X)_{\mathbb{Z}}$, and we make this identification henceforth. The first property allows us to extend the volume to $N_1(X)_{\mathbb{Q}}$ in a natural way, so that we can discuss its analytic properties:

Theorem 3.9 ([Laz04] Theorem 2.2.44, [LM09] Corollary C, [BFJ09] Theorem A). *Let X be a projective variety of dimension n.*
- *The volume extends to a continuous function on all of $N^1(X)$.*

- *The volume is continuously differentiable on the interior of $\overline{\mathrm{Eff}}^1(X)$ with*

$$\frac{d}{dt}\bigg|_{t=0} \widehat{\mathrm{vol}}(L + tE) = n\langle L^{n-1}\rangle \cdot E.$$

In fact, the definition of the volume (as an asymptotic limit of sections) makes sense for arbitrary $\mathbb{R}$-divisors, and the continuous extension as in Theorem 3.9 agrees with this naive definition ([FLK15]). Theorem 3.9 is optimal in that the volume fails to be $\mathcal{C}^2$ even in very simple examples, such as the blow-up of $\mathbb{P}^2$ at a point.

Remark 3.10. We see that the volume function for big divisors shares many of the desirable properties of the volume function for ample divisors. This is no accident: for an arbitrary big divisor L, one can view the volume as a "perturbation" of the self-intersection which removes extra contributions arising from the base locus. This viewpoint is made precise by the Fujita approximation theorem $\mathrm{vol}(L) = \langle L^n\rangle$.

However, there are also fundamental differences between the volume on the big cone and on the ample cone. [Cut86] constructs Cartier divisors whose volume is irrational, [BKS04] constructs varieties whose volume functions are not locally polynomial, and [KLM13] gives an example where the volume function is transcendental on a region of the big cone.

3.2. Convexity.

Theorem 3.11 ([Laz04] Theorem 11.4.9). *Let X be a projective variety of dimension n. The volume function is n-concave on the big cone: for big divisors L, D, we have*

$$\mathrm{vol}(L + D)^{1/n} \geq \mathrm{vol}(L)^{1/n} + \mathrm{vol}(D)^{1/n}.$$

For any concave function it is important to understand when strict concavity fails. By analogy with the surface case, we might look for a "Zariski decomposition" of divisors on varieties of arbitrary dimension. There are many different such notions in the literature (see [Pro03] for a survey) – however, most of these are equivalent for big divisors. The version we use here is due to Nakayama.

Usually a "Zariski decomposition theory" for a divisor L has two parts. We will only focus on the first: the removal of the divisorial stable base locus of L. (The second part is to study the behavior of the divisorial stable base locus upon blowing-up.)

Definition 3.12 ([Nak04]). Let X be a smooth projective variety. For a big divisor L and a prime divisor Γ, we define

$$\sigma_\Gamma(L) = \max\{c \in \mathbb{R}_{\geq 0} | L' \geq c\Gamma \text{ for every effective } \mathbb{R}\text{-divisor } L' \equiv L\}.$$

We then set $N_\sigma(L) = \sum_\Gamma \sigma_\Gamma(L)\Gamma$ and $P_\sigma(L) = L - N_\sigma(L)$.

We call the expression $L = P_\sigma(L) + N_\sigma(L)$ the Nakayama-Zariski decomposition of L; $P_\sigma(L)$ is the "positive part" and $N_\sigma(L)$ is the "negative part".

It turns out that the numerical class of $P_\sigma(L)$ is the same as the positive product $\langle L \rangle$. We can now quantify the failure of strict n-concavity of the volume function.

Theorem 3.13 ([LX15b] Theorem 1.6). *Let X be a smooth projective variety of dimension n. For big divisors L, D, we have $\mathrm{vol}(L + D)^{1/n} = \mathrm{vol}(L)^{1/n} + \mathrm{vol}(D)^{1/n}$ if and only if the numerical classes of $P_\sigma(L)$ and $P_\sigma(D)$ are proportional.*

In fact, the volume function for divisors fits into the abstract convexity framework formulated in [LX15a]. A posteriori, this motivates many of the well-known results in divisor theory (such as the Khovanskii-Teissier inequalities and the Morse criterion for bigness).

4. Volume for curves: convexity

[Xia15] shows that the polar transform can be used to define an interesting positivity function for curves. To obtain a function on $\overline{\mathrm{Eff}}_1(X)$, we need to take the polar transform of a function on the dual cone $\mathrm{Nef}^1(X)$.

Definition 4.1 ([Xia15]). Let X be a projective variety of dimension n and let α be a pseudo-effective curve class on X. We define the volume of α to be

$$\widehat{\mathrm{vol}}(\alpha) = \inf_{A \text{ ample}} \left(\frac{A \cdot \alpha}{\mathrm{vol}(A)^{1/n}} \right)^{\frac{n}{n-1}}.$$

We define the volume of α to be 0 outside of the pseudo-effective cone.

Note that α is an $\frac{n}{n-1}$-homogeneous function. In this section we study the properties of $\widehat{\mathrm{vol}}$.

4.1. **Zariski decompositions.** Although the volume function is differentiable on the interior of the nef cone, it does not vanish along the boundary. Thus the polar transform $\widehat{\mathrm{vol}}$ will fail to be strictly convex. Just as with divisors, we can capture the failure of concavity by a "Zariski decomposition" structure.

Definition 4.2 ([LX15a]). Let X be a projective variety of dimension n and let α be a big curve class on X. Then a Zariski decomposition for α is an expression

$$\alpha = B^{n-1} + \gamma$$

where B is a big and nef $\mathbb{R}$-Cartier divisor class, γ is pseudo-effective, and $B \cdot \gamma = 0$. We call B^{n-1} the "positive part" and γ the "negative part" of the decomposition.

Note that Definition 4.2 exactly generalizes Zariski's original construction as phrased in Definition 3.6. It turns out that Zariski decompositions always exist and that they exactly capture the failure of strict log concavity:

Theorem 4.3 ([LX15a] Theorem 1.2). *Every big curve class admits a unique Zariski decomposition.*

Theorem 4.4 ([LX15a] Theorem 1.5). *Let X be a projective variety of dimension n and let α and β be big curve classes on X. Then*

$$\widehat{\mathrm{vol}}(\alpha + \beta)^{n-1/n} \geq \widehat{\mathrm{vol}}(\alpha)^{n-1/n} + \widehat{\mathrm{vol}}(\beta)^{n-1/n}$$

with equality if and only if the positive parts B_α^{n-1} and B_β^{n-1} are proportional.

We define the complete intersection cone $\mathrm{CI}_1(X)$ to be the closure of the *set* of classes of the form H^{n-1} for an ample divisor H. Thus the positive part of the Zariski decomposition takes values in $\mathrm{CI}_1(X)$. It is important to emphasize that, in contrast to most cones in birational geometry, $\mathrm{CI}_1(X)$ need not be convex. However, it still has an easily-understood structure along the boundary ([LX15a]).

4.2. **Basic properties.** The basic properties of $\widehat{\mathrm{vol}}$ follow from formal properties of polar transforms.

Theorem 4.5 ([LX15a] Theorems 5.2 and 5.11). *Let X be a projective variety of dimension n.*

- *The volume is a continuous function on all of $N_1(X)$.*
- *The volume is continuously differentiable on $\mathrm{Big}_1(X)$, with*

$$\left.\frac{d}{dt}\right|_{t=0} \widehat{\mathrm{vol}}(\alpha + t\beta) = \frac{n}{n-1} B_\alpha \cdot \beta.$$

Note that B_α plays the role of the positive product in the derivative formula for divisors. A key advantage of this "positive product" for curves is that we do not need to pass to a birational model to define B_α – indeed, this feature is exactly what makes the divisor theory on surfaces so well behaved.

4.3. **Examples.**

Example 4.6 (Toric varieties). Let X be a simplicial projective toric variety of dimension n defined by a fan Σ. Suppose that the curve class α lies in the interior of the movable cone of curves, or equivalently, α is defined by a positive Minkowski weight on the rays of Σ. A classical theorem of Minkowski attaches to such a weight a polytope P_α whose facet normals are the rays of Σ and whose facet volumes are determined by the weights.

In this setting, the volume is calculated by an mixed volume problem: fixing P_α, amongst all polytopes whose normal fan refines Σ there is a unique Q (up to homothety) minimizing the mixed volume calculation

$$\left(\frac{V(P_\alpha^{n-1}, Q)}{\mathrm{vol}(Q)^{1/n}}\right)^{n/n-1}.$$

Then the volume is $n!$ times this minimal mixed volume, and the positive part of α is proportional to the $(n-1)$-product of the big and nef divisor defined by Q.

Note that if we let Q vary over all polytopes then the Brunn-Minkowski inequality shows that the minimum is given by $Q = cP_\alpha$, but the normal fan condition on Q yields a new version of this classical problem. This is the rare isoperimetric problem which can actually be solved explicitly, via the Zariski decomposition. (We expect this particular problem to admit a solution via a natural polytope construction that has yet to be discovered.)

Example 4.7 (Hyperkähler manifolds). Let X denote a hyperkähler manifold of dimension n and let q denote the (normalized) Beauville-Bogomolov form on $N^1(X)$, or via duality, on $N_1(X)$. The results of [Bou04, Section 4] and [Huy99] show that the volume and Nakayama-Zariski decomposition of divisors satisfy a natural compatibility with the Beauville-Bogomolov form:

  (1) The cone of movable divisors is q-dual to the cone of pseudo-effective divisors.
  (2) If P is a movable divisor then $\mathrm{vol}(P) = q(P,P)^{n/2}$.
  (3) Suppose that B is a big divisor and write $B = P + E$ for its Nakayama-Zariski decomposition. Then $q(P,E) = 0$ and if E is non-zero then $q(E,E) < 0$.

Exactly the analogous statements are true for curves by [LX15a]. We give the projective statements, but everything is true in the Kähler setting as well.

  (1) The cone of complete intersection curves is q-dual to the cone of pseudo-effective curves.
  (2) If α is a complete intersection curve class then $\widehat{\mathrm{vol}}(\alpha) = q(\alpha,\alpha)^{n/2(n-1)}$.
  (3) Suppose α is a big curve class and write $\alpha = B^{n-1} + \gamma$ for its Zariski decomposition. Then $q(B^{n-1}, \gamma) = 0$ and if γ is non-zero then $q(\gamma, \gamma) < 0$.

5. Volume for cycles: enumerative geometry

We now return to the problem of defining a volume function for cycles of arbitrary codimension. The main obstacle is that there is no obvious analogue of a "section ring". In particular, there does not seem to be a natural way to associate a vector bundle to a k-cycle.

We will need a change in perspective. Suppose we are given a linear series $\mathbb{P}H^0(X,L)$ on a projective variety X. For a general point $x \in X$, the divisors containing x are parametrized by a codimension 1 hyperplane in $\mathbb{P}H^0(X,L)$. Furthermore, if we fix several points in general position the corresponding hyperplanes intersect transversally. In sum: an element of the linear series can go through any $\dim H^0(X,L) - 1$ general points of X, but no more.

In this way we can interpret the volume as an asymptotic count of general points in divisors. An influential conjecture of [DELV11] suggests that

this approach generalizes naturally to cycles of arbitrary codimension. This suggestion works surprisingly well, even beyond the material we discuss here.

Below we give two different constructions of "volume-type" functions arising from this viewpoint. The first is easier to define; the second is easier to compute. Currently it is not clear which of the two is the better choice (or even whether the two functions are different).

5.1. **Mobility.**

Definition 5.1 ([Leh15]). Let X be a projective variety of dimension n and let $\alpha \in N_k(X)_{\mathbb{Z}}$ for $0 \le k < n$. The mobility count of α is

$$\mathrm{mc}(\alpha) = \max \left\{ b \in \mathbb{Z}_{\ge 0} \,\middle|\, \begin{array}{c} \text{any } b \text{ general points of } X \text{ are contained} \\ \text{in an effective cycle of class } \alpha \end{array} \right\}.$$

We point out two important features. First, we define the mobility count for a numerical class, and not a cycle. This is to ensure we obtain a numerical invariant at the end – one could equally well use rational or algebraic equivalence, but it is not clear whether the resulting definition would be the same. Second, we can define the mobility count for varieties with arbitrary singularities. This is a new feature which is useful even for divisors.

Just as for divisors, we want to calculate the asymptotic behavior of $m\alpha$ as m goes to infinity. To predict the growth rate, we use a heuristic argument on $\mathbb{P}^n$. Recall that a degree d divisor on projective space can contain approximately $d^n/n!$ general points. If we intersect $n - k$ such divisors, we obtain a degree d^{n-k} cycle of dimension k on projective space containing approximately $d^n/n!$ general points. (Each time we cut down by a hyperplane we lose 1 general points worth of flexibility, but this effect is negligible so long as $d \gg n$.) Rescaling, we expect a cycle of degree m to go through $\approx Cm^{n/n-k}$ general points of $\mathbb{P}^n$.

Lemma 5.2 ([Leh15] Proposition 5.1). *There is a constant $C = C(X, \alpha)$ such that $\mathrm{mc}(m\alpha) < Cm^{n/n-k}$ for every positive integer m.*

This "expected" growth rate is achieved by any complete intersection of ample divisors. The mobility function captures the leading exponent:

Definition 5.3 ([Leh15]). Let X be a projective variety of dimension n and suppose $\alpha \in N_k(X)_{\mathbb{Z}}$ for $0 \le k < n$. The mobility of α is

$$\mathrm{mob}(\alpha) = \limsup_{m \to \infty} \frac{\mathrm{mc}(m\alpha)}{m^{\frac{n}{n-k}}/n!}$$

For a Cartier divisor on a smooth variety, this definition coheres with the notion of volume defined above. On a singular variety X, [FL13, Theorem 1.16] shows that the mobility of a Weil divisor class α is the same as the maximum of the volumes of all preimages of α on resolutions of X.

Theorem 5.4 ([Leh15] Theorem 1.2). *Let X be a projective variety of dimension n. Then* mob *extends to a continuous $\frac{n}{n-k}$-homogeneous function on all of $N_k(X)$.*

In particular, the mobility is positive precisely for big classes. This gives a geometric interpretation of boundary classes of $\overline{\mathrm{Eff}}_k(X)$; the boundary consists of those classes for which the asymptotic point counts do not grow too quickly.

Sketch of proof: Fix a very ample divisor H and a class $\alpha \in N_k(X)_{\mathbb{Z}}$. For convenience set $b = \mathrm{mc}(\alpha)$. We prove the following version of our statement: choose an integer s so that $sH^n \geq \alpha \cdot H^{n-k}$. If α is not big, then there is a constant $\epsilon = \epsilon(n,k) > 0$ such that

$$b \leq C(n,k)H^n s^{\frac{n}{n-k}-\epsilon}$$

Let A be a general element of $|\lceil s^{1/n-k} \rceil H|$. Consider specializing b general points in X to b general points on A. For each configuration of points there will be a cycle of class $m\alpha$ containing the points, and we can consider the limit cycle Z containing b general points of A. We separate Z into two pieces: the components Z_A contained in A and the components Z_B not contained in A.

We now bound the number of points contained in each piece by induction. The intersection of Z_B with A is a $(k-1)$-cycle on A of class $\alpha \cdot A$; by induction, it can go through at most

$$C(n-1,k-1)H|_A^{n-1} \cdot s^{\frac{n-1}{n-k}} = C(n-1,k-1)H^n s^{\frac{n}{n-k}}$$

general points of A. Z_A is a k-cycle on A whose pushforward to X is less pseudo-effective than α. An easy lemma shows that Z_A can not be big on A. Note also that we can decrease s in this situation, since

$$Z_A \cdot H|_A^{n-k} \leq \alpha \cdot H^{n-k} \leq s^{\frac{n-k-1}{n-k}} H|_A^{n-1}.$$

Thus (after rounding to ensure our new s-value is an integer) Z_A can contain at most

$$C(n-1,k)H|_A^{n-1}(s^{\frac{n-k-1}{n-k}})^{\frac{n-1}{n-k-1}-\epsilon(n-1,k)} = C(n-1,k)H^n s^{\frac{n}{n-k}-\epsilon(n,k)}$$

We can bound b by adding these two expressions.

Note that we have not quite proved the statement, since the first contribution grows too quickly. We can compensate by slightly decreasing the degree of A. The exponent of s in the first equation will decrease slightly and the exponent of s in the second equation will increase slightly, yielding a bound as claimed. $\qquad\square$

It is not yet known whether the mobility is differentiable or log-concave. In fact, it is not even clear whether one should expect log concavity to hold: for divisors concavity is deduced from the Hodge Index Theorem, but the signature of the intersection form in intermediate dimensions can be different.

The mobility is quite difficult to compute. Nevertheless, the following compelling example hints at a broader picture that has not yet been uncovered.

Example 5.5. Let ℓ denote the class of a line on $\mathbb{P}^3$. The mobility of ℓ is determined by an enumerative question: what is the minimal degree of a curve in $\mathbb{P}^3$ going through b general points? The answer to this question is not known, even in an asymptotic sense. More precisely, we know that the minimal degree is $\approx Cb^{2/3}$ for some constant C, but we do not know the value of C.

[Per87] conjectures that the "optimal" curves (which maximize the mobility count in terms of their degree) are complete intersection curves. This would easily imply that $\mathrm{mob}(\ell) = 1$. [Per87] analyzes this question in detail in low degrees. In general, it is easy to prove a lower bound $C \geq 1$ and [Per87] shows that for families of smooth curves we have $C \geq 2^{2/3}$, but this is still far from the upper bound (and expected value) $C \leq 6^{2/3}$.

The most surprising, and appealing, feature of Perrin's conjecture is that the complete intersection curves are "picked out" from the viewpoint of enumerative geometry. We expect a similar statement to hold in general.

Conjecture 5.6 ([Leh15] Question 6.1). *Let X be a projective variety of dimension n and let H be an ample divisor on X. Then for $0 < k < n$ we have $\mathrm{mob}(H^{n-k}) = \mathrm{vol}(H)$.*

5.1.1. *Zariski decompositions.* [FL13] shows that the mobility admits a weak type of "Zariski decomposition structure".

Definition 5.7 ([FL13]). Let X be a projective variety and let $\alpha \in N_k(X)$ be a big class. A weak Zariski decomposition for α is an expression $\alpha = \beta + \gamma$ where $\beta \in \mathrm{Mov}_k(X)$, $\gamma \in \overline{\mathrm{Eff}}_k(X)$, and $\mathrm{mob}(\alpha) = \mathrm{mob}(\beta)$.

Theorem 5.8 ([FL13] Theorem 1.6). *Every big cycle class admits a weak Zariski decomposition.*

The construction is weaker than the version for curves described above since it provides less information about the positive part (often we have $\mathrm{CI}_1(X) \subsetneq \mathrm{Mov}_1(X)$). However, it is just the right tool for understanding the behavior of mobility under birational maps.

Theorem 5.9 ([FL13] Proposition 6.11). *Let $\pi : Y \to X$ be a birational morphism of projective varieties. Let $\alpha \in N_k(X)$. Then*

$$\mathrm{mob}(\alpha) = \sup_{\beta \in N_k(Y), \pi_* \beta = \alpha} \mathrm{mob}(\beta)$$

and the supremum on the right hand side is achieved by a class β.

5.2. **Weighted mobility.** We now define the second volume-type function arising from enumerative geometry. Following a suggestion of R. Lazarsfeld,

we define the weighted mobility count of a class $\alpha \in N_k(X)_{\mathbb{Z}}$ as:

$$\text{wmc}(\alpha) = \max \left\{ b \in \mathbb{Z}_{\geq 0} \;\middle|\; \begin{array}{c} \text{there is a } \mu \in \mathbb{Z}_{>0} \text{ and an effective cycle of} \\ \text{class } \mu\alpha \text{ through any } b \text{ points of } X \text{ with} \\ \text{multiplicity at least } \mu \text{ at each point} \end{array} \right\}.$$

This definition has the effect of counting singular points with a higher "weight". This convention better reflects the intersection theory on the blow-up of the points, as the strict transform of a cycle which is singular at a point will be have larger intersection against the exceptional divisor than the strict transform of a smooth cycle. In particular, the weighted mobility is closely related to the computation of Seshadri constants.

The expected growth rate of the mobility count on a variety of dimension n is still $\text{wmc}(m\alpha) \sim Cm^{n/n-k}$, suggesting:

Definition 5.10 ([Leh15]). Let X be an integral projective variety of dimension n and suppose $\alpha \in N_k(X)_{\mathbb{Z}}$ for $0 \leq k < n$. The weighted mobility of α is

$$\text{wmob}(\alpha) = \limsup_{m \to \infty} \frac{\text{wmc}(m\alpha)}{m^{\frac{n}{n-k}}}$$

The rescaling factor $n!$ is now omitted to ensure that the hyperplane class on $\mathbb{P}^n$ has weighted mobility 1.

Theorem 5.11 ([Leh15] Theorem 1.12). *Let X be an integral projective variety. Then* wmob *extends to a continuous $\frac{n}{n-k}$-homogeneous function on $N_k(X)$. In particular, $\alpha \in N_k(X)$ is big if and only if* $\text{wmob}(\alpha) > 0$.

The advantage of the weighted mobility is that it can be calculated using Seshadri constants.

Example 5.12 ([Leh15] Example 8.19). Let X be a projective variety of dimension n and let H be an ample divisor on X. Then for $0 < k < n$ we have $\text{wmob}(H^{n-k}) = \text{vol}(H)$. The "hard" inequality $\leq$ is proved by an intersection-theoretic calculation on blow-ups of X.

6. Comparison of volumes for curves

We have defined several different volume-type functions for curves, and it is interesting to ask how they compare. Surprisingly, the construction arising from abstract convex duality matches up with the asymptotic point counts.

Theorem 6.1 ([LX15b] Theorem 1.3). *Let X be a smooth projective variety of dimension n and let $\alpha \in \overline{\text{Eff}}_1(X)$ be a pseudo-effective curve class. Then:*

(1) $\widehat{\text{vol}}(\alpha) = \text{wmob}(\alpha)$.
(2) $\widehat{\text{vol}}(\alpha) \leq \text{mob}(\alpha) \leq n!\widehat{\text{vol}}(\alpha)$.
(3) *Assume Conjecture 5.6 for curve classes. Then* $\text{mob}(\alpha) = \widehat{\text{vol}}(\alpha)$.

Sketch of proof: The proof relies on the following ingredients:

- The Zariski decompositions for mob and $\widehat{\text{vol}}$.
- An analysis of the birational behavior of volume functions.
- Classical techniques for constructing divisors with imposed singularities at given points.

We sketch the proof of Theorem 6.1.(3), which is the most difficult part. [FL13, Corollary 6.16] shows that for any big curve class α, there is a big and nef curve class β satisfying:

- $\text{mob}(\alpha) = \text{mob}(\beta)$.
- $\text{mob}(\phi^*\beta) = \text{mob}(\beta)$ for any birational map ϕ.

The most subtle part of the proof is to show that there is a sequence of birational maps ϕ such that $\phi^*\beta$ becomes "closer and closer" to a complete intersection class. Then since we are assuming Conjecture 5.6, we see that we can approximate $\text{mob}(\phi^*\beta) = \text{mob}(\beta)$ using intersection theory.

Using the birational invariance of mob and the inequality $\widehat{\text{vol}} \leq \text{mob}$, we are able to deduce that $\widehat{\text{vol}}(\phi^*\beta) = \widehat{\text{vol}}(\beta)$ for any birational map. However, an additional lemma shows that this property can only hold when β is a complete intersection class. Thus the Zariski decompositions for $\widehat{\text{vol}}$ and mob coincide, and the desired conclusion follows easily. $\square$

7. An alternative dual function

We close with a different application of the polar transform. Recall that the polar transform relates positive functions on dual cones. Instead of using the duality $\overline{\text{Eff}}_1(X) \leftrightarrow \text{Nef}^1(X)$, we could use the duality $\text{Mov}_1(X) \leftrightarrow \overline{\text{Eff}}^1(X)$ (as proved by [BDPP13]).

Definition 7.1 ([Xia15]). Let X be a projective variety of dimension n. For any curve class $\alpha \in \text{Mov}_1(X)$ define

$$\mathfrak{M}(\alpha) = \inf_{L \text{ big divisor class}} \left(\frac{L \cdot \alpha}{\text{vol}(L)^{1/n}} \right)^{n/n-1}.$$

The main structure theorem for $\mathfrak{M}$ relies on a refined version of a theorem of [BDPP13] describing the movable cone of curves. In [BDPP13], it is proved that the movable cone $\text{Mov}_1(X)$ is the closure of the cone *generated* by $(n-1)$-self positive products of big divisors. We show that $\text{Mov}_1(X)$ is the closure of the *set* of $(n-1)$-self positive products of big divisors.

Theorem 7.2 ([LX15b] Theorem 1.8). *Let X be a smooth projective variety of dimension n and let α be an interior point of $\text{Mov}_1(X)$. Then there is a unique big movable divisor class L_α lying in the interior of $\text{Mov}^1(X)$ and depending continuously on α such that $\langle L_\alpha^{n-1} \rangle = \alpha$.*
  We have $\mathfrak{M}(\alpha) = \text{vol}(L_\alpha)$.

Thus, $\mathfrak{M}$ can be used to identify the "$(n-1)$st-root" of a class in the interior of $\text{Mov}^1(X)$. It turns out that this description extends naturally to

the boundary of $\mathrm{Mov}_1(X)$ as well; see [LX15b, Theorem 1.9] for details. As an interesting corollary of Theorem 7.2, we obtain:

Corollary 7.3 ([LX15b] Corollary 1.10). *Let X be a projective variety of dimension n. Then the rays over classes of irreducible curves which deform to dominate X are dense in $\mathrm{Mov}_1(X)$.*

We expect that every $\mathbb{Q}$-ray in the interior of $\mathrm{Mov}_1(X)$ should be generated by an irreducible moving curve.

Just as for the volume function, $\mathfrak{M}$ admits an enumerative interpretation. We can define mob_{mov} and wmob_{mov} for curve classes analogously to mob and wmob, except that we only count points on families of curves whose general member is a sum of irreducible movable curves. Note that this function can only be positive on the movable cone of curves.

Theorem 7.4 ([LX15b] Theorem 7.3). *Let X be a smooth projective variety of dimension n and let $\alpha \in \mathrm{Mov}_1(X)^\circ$. Then:*

(1) $\mathfrak{M}(\alpha) = \mathrm{wmob}_{mov}(\alpha)$.
(2) Assume Conjecture 5.6 for curve classes. Then $\mathfrak{M}(\alpha) = \mathrm{mob}_{mov}(\alpha)$.

Example 7.5 (Toric varieties). Suppose that X is a smooth toric variety of dimension n defined by a fan Σ in a finite rank free abelian group N. Let r_i denote the rays of the fan.

A class α in the interior of the movable cone corresponds to a Minkowski weight which is positive on the rays r_i. A classical theorem of Minkowski associates to α a polytope P_α whose facet normals are rays of Σ and whose facet volumes are determined by the weights. Then P_α corresponds to the big movable divisor L satisfying $\langle L^{n-1} \rangle = \alpha$, and $\mathfrak{M}(\alpha) = n!\, \mathrm{vol}(P_\alpha)$.

Example 7.6 (Mori dream spaces). The cone of movable divisors of a Mori dream space X admits a chamber decomposition reflecting the structure of all small $\mathbb{Q}$-factorial modifications of X ([HK00]). It is interesting to ask for an analogous structure for curves. Note that if we simply take the duals of the chamber decomposition of divisors we do not remain in the pseudo-effective cone, suggesting a different approach is needed.

[LX15b] shows that the movable cone of curves admits a similar chamber decomposition, arising from the complete intersection cones on birational models. The main difference from the divisor version is that the chambers are non-convex cones; however, otherwise the analogous properties hold. (Instead of dualizing, this construction passes the chamber structure from divisors to curves via the map $\langle -^{n-1} \rangle$.)

The two functions $\widehat{\mathrm{vol}}$ and $\mathfrak{M}$ give a good way of working with this chamber structure. On the movable cone of curves we have an inequality $\widehat{\mathrm{vol}} \geq \mathfrak{M}$, and equality is achieved precisely on the complete intersection cone of X. As we pass to different models $\mathfrak{M}$ is preserved while $\widehat{\mathrm{vol}}$ changes as we vary the ample cone.

References

[AS95]   Urban Angehrn and Yum Tong Siu, *Effective freeness and point separation for adjoint bundles*, Invent. Math. **122** (1995), no. 2, 291–308.

[BDPP13]   Sébastien Boucksom, Jean-Pierre Demailly, Mihai Păun, and Thomas Peternell, *The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension*, J. Algebraic Geom. **22** (2013), no. 2, 201–248.

[BFJ09]   Sébastien Boucksom, Charles Favre, and Mattias Jonsson, *Differentiability of volumes of divisors and a problem of Teissier*, J. Algebraic Geom. **18** (2009), no. 2, 279–308.

[BKS04]   Thomas Bauer, Alex Küronya, and Tomasz Szemberg, *Zariski chambers, volumes, and stable base loci*, J. Reine Angew. Math. **576** (2004), 209–233.

[Bou02a]   Sébastien Boucksom, *Cônes positifs des variétés complexes compactes*, 2002, Grenoble Thesis.

[Bou02b]   _____, *On the volume of a line bundle*, Internat. J. Math. **13** (2002), no. 10, 1043–1063.

[Bou04]   _____, *Divisorial Zariski decompositions on compact complex manifolds*, Ann. Sci. École Norm. Sup. (4) **37** (2004), no. 1, 45–76.

[Cut86]   Steven Dale Cutkosky, *Zariski decomposition of divisors on algebraic varieties*, Duke Math. J. **53** (1986), no. 1, 149–156.

[DEL00]   Jean-Pierre Demailly, Lawrence Ein, and Robert Lazarsfeld, *A subadditivity property of multiplier ideals*, Michigan Math. J. **48** (2000), 137–156, Dedicated to William Fulton on the occasion of his 60th birthday.

[DELV11]   Olivier Debarre, Lawrence Ein, Robert Lazarsfeld, and Claire Voisin, *Pseudoeffective and nef classes on abelian varieties*, Compos. Math. **147** (2011), no. 6, 1793–1818.

[Dem93]   Jean-Pierre Demailly, *A numerical criterion for very ample line bundles*, J. Differential Geom. **37** (1993), no. 2, 323–374.

[ELM$^+$05] Lawrence Ein, Robert Lazarsfeld, Mircea Mustaţă, Michael Nakamaye, and Mihnea Popa, *Asymptotic invariants of line bundles*, Pure Appl. Math. Q. **1** (2005), no. 2, Special Issue: In memory of Armand Borel. Part 1, 379–403.

[ELN96]   Lawrence Ein, Robert Lazarsfeld, and Michael Nakamaye, *Zero-estimates, intersection theory, and a theorem of Demailly*, Higher-dimensional complex varieties (Trento, 1994), de Gruyter, Berlin, 1996, pp. 183–207.

[FL13]   Mihai Fulger and Brian Lehmann, *Zariski decompositions of numerical cycle classes*, 2013, arXiv:1310.0538, to appear in J. Alg. Geom.

[FL15]   _____, *Positive cones of dual cycle classes*, 2015, arXiv:1408.5154 [math.AG], to appear in Alg. Geom.

[FLK15]   Mihai Fulger, Brian Lehmann, and János Kollár, *Volume and hilbert function of r-divisors*, 2015, arXiv:1309.0880, to appear in Mich. Math. J.

[FO16]   Kento Fujita and Yuji Odaka, *On the K-stability of fano varieties and anticanonical divisors*, 2016, arXiv:1602.01305 [math.AG].

[Fuj79]   Takao Fujita, *On Zariski problem*, Proc. Japan Acad. Ser. A Math. Sci. **55** (1979), no. 3, 106–110. MR 531454 (80j:14029)

[Fuj94]   _____, *Approximating Zariski decomposition of big line bundles*, Kodai Math. J. **17** (1994), no. 1, 1–3.

[Fuj15a]   Kento Fujita, *On the K-stability of fano varieties and anticanonical divisors*, 2015, arXiv:1508.04052 [math.AG].

[Fuj15b]   _____, *Optimal bounds for the volumes of Kähler-Einstein Fano manifolds*, 2015, arXiv:1508.04578 [math.AG].

[Ful84]   William Fulton, *Intersection theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 2, Springer-Verlag, Berlin, 1984.

[HK00]   Yi Hu and Sean Keel, *Mori dream spaces and GIT*, Michigan Math. J. **48** (2000), 331–348, Dedicated to William Fulton on the occasion of his 60th birthday. MR 1786494

[HM06]   Christopher D. Hacon and James McKernan, *Boundedness of pluricanonical maps of varieties of general type*, Invent. Math. **166** (2006), no. 1, 1–25.

[HMX13]   Christopher D. Hacon, James McKernan, and Chenyang Xu, *On the birational automorphisms of varieties of general type*, Ann. of Math. (2) **177** (2013), no. 3, 1077–1111.

[Huy99]   Daniel Huybrechts, *Compact hyperkähler manifolds: basic results*, Inventiones mathematicae **135** (1999), no. 1, 63–113.

[Kho89]   Askold G. Khovanskiĭ, *Newton polytopes (algebra and geometry)*, Theory of operators in function spaces (Russian) (Kuybyshev, 1988), Saratov. Gos. Univ., Kuĭbyshev. Filial, Kuybyshev, 1989, pp. 202–221.

[KK12]   Kiumars Kaveh and Askold G. Khovanskii, *Newton-Okounkov bodies, semigroups of integral points, graded algebras and intersection theory*, Ann. of Math. (2) **176** (2012), no. 2, 925–978.

[KLM13]   Alex Küronya, Victor Lozovanu, and Catriona Maclean, *Volume functions of linear series*, Math. Ann. **356** (2013), no. 2, 635–652.

[Laz04]   Robert Lazarsfeld, *Positivity in algebraic geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 48, Springer-Verlag, Berlin, 2004, Classical setting: line bundles and linear series.

[Leh15]   Brian Lehmann, *Volume-type functions for numerical cycle classes*, 2015, arXiv:1309.0880 [math.AG], to appear in Duke Math. J.

[LM09]   Robert Lazarsfeld and Mircea Mustaţă, *Convex bodies associated to linear series*, Ann. Sci. Éc. Norm. Supér. (4) **42** (2009), no. 5, 783–835.

[LX15a]   Brian Lehmann and Jian Xiao, *Convexity and Zariski decomposition structure*, 2015, https://www2.bc.edu/brian-lehmann/papers/cnvxzar.pdf.

[LX15b]   ———, *Positivity functions for curves on algebraic varieties*, 2015, https://www2.bc.edu/brian-lehmann/papers/poscurves.pdf.

[MR15]   David McKinnon and Mike Roth, *Seshadri constants, diophantine approximation, and Roth's theorem for arbitrary varieties*, Invent. Math. **200** (2015), no. 2, 513–583. MR 3338009

[Nak04]   Noboru Nakayama, *Zariski-decomposition and abundance*, MSJ Memoirs, vol. 14, Mathematical Society of Japan, Tokyo, 2004.

[Per87]   Daniel Perrin, *Courbes passant par m points généraux de $\mathbb{P}^3$*, Mém. Soc. Math. France (N.S.) (1987), no. 28-29, 138.

[Pro03]   Yuri G. Prokhorov, *On the Zariski decomposition problem*, Tr. Mat. Inst. Steklova **240** (2003), no. Biratsion. Geom. Linein. Sist. Konechno Porozhdennye Algebry, 43–72, translation in Proc. Steklov Inst. Math. 2003, no. 1 (240), 37–65.

[Roc70]   R. Tyrrell Rockafellar, *Convex analysis*, Princeton Mathematical Series, No. 28, Princeton University Press, Princeton, N.J., 1970.

[Tak06]   Shigeharu Takayama, *Pluricanonical systems on algebraic varieties of general type*, Invent. Math. **165** (2006), no. 3, 551–587.

[Tei79]   Bernard Teissier, *Du théoreme de l'index de Hodge aux inégalités isopérimétriques*, CR Acad. Sci. Paris Sér. AB **288** (1979), no. 4, A287–A289.

[Tei82]   ———, *Bonnesen-type inequalities in algebraic geometry. I. Introduction to the problem*, Seminar on Differential Geometry, Ann. of Math. Stud., vol. 102, Princeton Univ. Press, Princeton, N.J., 1982, pp. 85–105.

[Tsu06]   Hajime Tsuji, *Pluricanonical systems of projective varieties of general type. I*, Osaka J. Math. **43** (2006), no. 4, 967–995.

[Xia15]   Jian Xiao, *Characterizing volume via cone duality*, 2015, arXiv:1502.06450.
[Yua08]   Xinyi Yuan, *Big line bundles over arithmetic varieties*, Invent. Math. **173** (2008), no. 3, 603–649.
[Yua09]   ———, *On volumes of arithmetic line bundles*, Compos. Math. **145** (2009), no. 6, 1447–1464.
[Zar62]   Oscar Zariski, *The theorem of Riemann-Roch for high multiples of an effective divisor on an algebraic surface*, Ann. of Math. (2) **76** (1962), 560–615.