THE VANISHING CONJECTURE FOR MAPS OF TOR, SPLINTERS, AND DERIVED SPLINTERS

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Abstract. This is an extended version of the lecture notes for the UIC homological conjecture workshop. We include a solution to Exercise 2.7, and add some further observation on the vanishing conjecture for maps of Tor in the appendix.

1. The vanishing conjecture for maps of Tor

In [HH95], Hochster and Huneke proposed the following vanishing conjecture for maps of Tor:

Conjecture 1.1. *Let A be a regular domain, let R be a module-finite and torsion-free extension of A, and let $R \to S$ be any homomorphism from R to a regular ring S. Then for every A-module M and every $i \geq 1$, the map $\operatorname{Tor}_i^A(M,R) \to \operatorname{Tor}_i^A(M,S)$ vanishes.*

This is proved by Hochster-Huneke when A, R, S all have equal characteristic [HH95, Theorem 4.1], using weakly functorial big Cohen-Macaulay algebras. Before we discuss the proof we first observe some applications that address the importance of Conjecture 1.1.

Proposition 1.2. *The vanishing conjecture for maps of Tor implies the monomial conjecture, and hence the direct summand conjecture.*

Proof. Let $(R,\mathfrak{m})$ be a complete local domain and let $x_1, \dots, x_d$ be a system of parameters of R (we may take $x_1 = p$ in mixed characteristic). If the monomial conjecture fails, then we have $x_1^t x_2^t \cdots x_d^t = r_1 x_1^{t+1} + \cdots + r_d x_d^{t+1}$ for some t and some $r_1, \dots, r_d \in R$. By Cohen's structure theorem, we have a module-finite extension $A \to R$ such that A is a complete regular local ring and $x_1, \dots, x_d$ is a system of parameters of A.

Let $M = A/(x_1^{t+1}, \dots, x_d^{t+1}, x_1^t x_2^t \cdots x_d^t)$ and $S = R/\mathfrak{m}$, the residue field of R. Then the vanishing conjecture for maps of Tor implies that the map

$$\operatorname{Tor}_1^A(A/(x_1^{t+1}, \dots, x_d^{t+1}, x_1^t x_2^t \cdots x_d^t), R) \to \operatorname{Tor}_1^A(A/(x_1^{t+1}, \dots, x_d^{t+1}, x_1^t x_2^t \cdots x_d^t), R/\mathfrak{m})$$

is the zero map. The minimal free resolution of M over A has the form: $\cdots \to A^{d+1} \to A \to 0$ where the free generators of A^{d+1} are mapped to $x_1^t x_2^t \cdots x_d^t, x_1^{t+1}, \dots, x_d^{t+1}$. The relation $(-1, r_1, \dots, r_d)$ represents an element of $\operatorname{Tor}_1^A(A/(x_1^{t+1}, \dots, x_d^{t+1}, x_1^t x_2^t \cdots x_d^t), R)$ and so we know it maps to 0 in $\operatorname{Tor}_A^1(A/(x_1^{t+1}, \dots, x_d^{t+1}, x_1^t x_2^t \cdots x_d^t), R/\mathfrak{m}) \cong (R/\mathfrak{m})^{d+1}$. But this is a contradiction because we have a unit in the relation $(1, r_1, \dots, r_d)$ in the first component. $\square$

Remark 1.3. In the above proof we apply Conjecture 1.1 to R mixed characteristic and $S = R/\mathfrak{m}$ a field (and thus equal characteristic). We will see later that even if we restrict ourselves to A, R, S all mixed characteristic, Conjecture 1.1 still implies the direct summand conjecture (Theorem 2.11).

From now on, we always assume A, R, S all have the same characteristic.

Proposition 1.4. *The vanishing conjecture for maps of Tor implies that direct summands of regular rings are Cohen-Macaulay.*

Proof. Let R be a direct summand of a regular ring S. We want to show that R is Cohen-Macaulay. The question is local on R, so we may replace R by its localization at a prime ideal P and replace S by S_P. Therefore we may assume $(R, \mathfrak{m})$ is local. Next we can take the completion of R with respect to $\mathfrak{m}$ and replace S by its completion at $\mathfrak{m}S$ (all the hypothesis are preserved by the exercises below). Also notice that $S = \prod S_i$ is a product of regular domains, if $(R, \mathfrak{m}) \to S$ splits then there exists i such that $R \to S_i$ splits. Therefore we can assume $(R, \mathfrak{m})$ is complete local, $R \to S$ splits, and S is a regular domain (and hence R is also a domain).

Now by Cohen's structure theorem, we have a module-finite extension $A \to R$ such that A is a complete regular local ring. Let $x_1, \ldots, x_d$ be a regular system of parameters of A. Now we apply the vanishing conjecture for maps of Tor to $M = A/(x_1, \ldots, x_d)$. We have

$$\mathrm{Tor}_i^A(A/(x_1, \ldots, x_d), R) \to \mathrm{Tor}_i^A(A/(x_1, \ldots, x_d), S)$$

vanishes for all $i \geq 1$. However, we also know that this map is injective because $R \to S$ is a split injection. Thus we have

$$\mathrm{Tor}_i^A(A/(x_1, \ldots, x_d), R) = H_i(x_1, \ldots, x_d, R) = 0$$

for all $i \geq 1$. This implies $x_1, \ldots, x_d$ is a regular sequence on R and hence R is Cohen-Macaulay. $\square$

Exercise 1.5. Prove that if S is regular and $J \subseteq S$, then the completion of S with respect to J is also regular.

Exercise 1.6. Prove that if $(R, \mathfrak{m}) \to S$ is split, then $\widehat{R} \to \widehat{S}^{\mathfrak{m}S}$ is also split.

Remark 1.7. In fact, Conjecture 1.1 implies that direct summand of regular rings are pseudo-rational. We refer to [Ma15, Proposition 3.4] for a more general result.

Our next goal is to prove the vanishing conjecture for maps of Tor using the weakly functorial big Cohen-Macaulay algebras. Recall that a (not necessarily finitely generated) $(R, \mathfrak{m})$-module M is big Cohen-Macaulay if one system of parameters of R is a regular sequence on M, it is called balanced big Cohen-Macaulay if every system of parameters of R is a regular sequence on M. A (balanced) big Cohen-Macaulay algebra is an R-algebra that is (balanced) big Cohen-Macaulay as an R-module.

We state the following remarkable result of Hochster-Huneke [HH95].

Theorem 1.8 (Existence of weakly functorial big Cohen-Macaulay algebras)**.** *We can assign to every equicharacteristic excellent local domain R a balanced big Cohen-Macaulay algebra B_R in such a way that if $R \to S$ is a local homomorphism of excellent local domains (of equicharacteristic), then the map extends to a map $B_R \to B_S$; i.e., there exists a commutative diagram:*

$$\begin{array}{ccc} B_R & \longrightarrow & B_S \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \end{array}$$

In fact, in characteristic $p > 0$, we can take $B_R = R^+$, the absolute integral closure of R, i.e., the integral closure of R inside an algebraic closure of its fraction field.

The proof of Theorem 1.8 in characteristic 0 is a delicate reduction to $p > 0$ argument, and the result in characteristic $p > 0$ was proved in [HH92]. A simpler proof that R^+ is balanced big Cohen-Macaulay in characteristic $p > 0$ was later found in [HL07], where it follows from the following crucial local cohomology annihilating result [HL07, Lemma 2.2].

Theorem 1.9. *Let $(R, \mathfrak{m})$ be a local domain of characteristic $p > 0$ that is a homomorphic image of a Gorenstein local ring. Let $N \subseteq H^j_{\mathfrak{m}}(R)$ be a Frobenius stable submodule of finite length. Then there exists a module-finite extension S of R such that the natural map $H^j_{\mathfrak{m}}(R) \to H^j_{\mathfrak{m}}(S)$ sends N to 0.*

In characteristic 0 or mixed characteristic, R^+ is not a balanced big Cohen-Macaulay algebra in general.

Exercise 1.10. Let $(R, \mathfrak{m})$ be a complete local domain. Prove the following:

(1) If R has characteristic 0, then R^+ is big Cohen-Macaulay if and only if $\dim R \leq 2$.
(2) If R has mixed characteristic, then R^+ is big Cohen-Macaulay if $\dim R \leq 2$, and is not big Cohen-Macaulay if $\dim R \geq 4$.

The only remaining open case about big Cohen-Macaulayness of R^+ is the following:

Question 1.11. Let $(R, \mathfrak{m})$ be a complete local domain of dimension three of mixed characteristic. Is R^+ balanced big Cohen-Macaulay?

We state a related interesting question of Lyubeznik:

Question 1.12. Let $(R, \mathfrak{m})$ be a complete local domain of mixed characteristic. Is $R^+/\sqrt{pR^+}$ balanced big Cohen-Macaulay over R/pR?

To prove Conjecture 1.1 in equal characteristic we need one more lemma:

Lemma 1.13. *Let $(A, \mathfrak{m})$ be a regular local ring. Then a (not necessarily finitely generated) A-module M is balanced big Cohen-Macaulay if and only if M is faithfully flat over A.*

Proof. If M is faithfully flat over A, then every system of parameters $(x_1, \dots, x_d)$ is a regular sequence in M (since they form a regular sequence on A and we know that $\mathfrak{m}M \neq M$).

Now suppose M is a balanced big Cohen-Macaulay A-module. We will show $\operatorname{Tor}_i^A(N, M) = 0$ for all finitely generated A-module N and every $i > 0$. We use descending induction. This is clearly true when $i > d$ since A has finite global dimension d. Suppose $\operatorname{Tor}_{k+1}^A(N, M) = 0$ for all N, we want to show $\operatorname{Tor}_k^A(N, M) = 0$ for all N. By considering a prime cyclic filtration of N, it is enough to prove this for $N = R/P$. Let $h = \operatorname{ht} P$. We can pick a regular sequence $x_1, \dots, x_h \in P$. Now P is an associated prime of $(x_1, \dots, x_h)$ so we have $0 \to R/P \to R/(x_1, \dots, x_h) \to C \to 0$. The long exact sequence for Tor gives:

$$\operatorname{Tor}_{k+1}^A(C, M) \to \operatorname{Tor}_k^A(R/P, M) \to \operatorname{Tor}_k^A(R/(x_1, \dots, x_h), M).$$

Now $\operatorname{Tor}_k^A(R/(x_1, \dots, x_h), M) = H_k(x_1, \dots, x_h, M) = 0$ because $x_1, \dots, x_h$ is a regular sequence on M, and $\operatorname{Tor}_{k+1}^A(C, M) = 0$ by induction. It follows that $\operatorname{Tor}_k^A(R/P, M) = 0$. $\square$

Proof of Conjecture 1.1 in equal characteristic. Clearly by a direct limit argument, we may assume M is a finitely generated R-module. If the map

$$\operatorname{Tor}_i^A(M, R) \to \operatorname{Tor}_i^A(M, S)$$

is nonzero then there exists a prime ideal P of S such that the map remains nonzero after localizing at P. We may thus assume S is local. Next we localize A, R, M at the prime ideal of A that is the contraction of the maximal ideal of S, and then replace A, M, R by their completions. Therefore, we may assume that A, S are both complete regular local rings. The kernel of the map $R \to S$ contains a minimal prime P of R. Since $A \to R$ is torsion-free, $A \to R/P$ is still a module-finite extension. Thus we have a factorization $A \to R \to R/P \to S$, which induces

$$\operatorname{Tor}_i^A(M, R) \to \operatorname{Tor}_i^A(M, R/P) \to \operatorname{Tor}_i^A(M, S),$$

thus it suffices to show the second map is zero. Hence by replacing R by R/P we may assume R is a complete local domain.

By Theorem 1.8, we have a commutative diagram:

$$\begin{tikzcd} B_R \arrow[r] & B_S \\ R \arrow[u] \arrow[r] & S \arrow[u] \end{tikzcd}$$

where B_R and B_S are balanced big Cohen-Macaulay algebras for R and S respectively. This induces commutative diagram:

$$\begin{tikzcd} \operatorname{Tor}_i^A(M, B_R) \arrow[r] & \operatorname{Tor}_i^A(M, B_S) \\ \operatorname{Tor}_i^A(M, R) \arrow[u] \arrow[r] & \operatorname{Tor}_i^A(M, S) \arrow[u] \end{tikzcd}$$

Since B_R is a balanced big Cohen-Macaulay algebra over R (and hence also over A), it is faithfully flat over A by Lemma 1.13 so $\operatorname{Tor}_i^A(M, B_R) = 0$ for all $i \geq 1$. Moreover, by Lemma 1.13 again B_S is faithfully flat over S since it is balanced big Cohen-Macaulay over S, thus $\operatorname{Tor}_i^A(M, S) \to \operatorname{Tor}_i^A(M, B_S)$ is injective. Chasing the diagram above we get that the map $\operatorname{Tor}_i^A(M, R) \to \operatorname{Tor}_i^A(M, S)$ vanishes for all $i \geq 1$. $\square$

Exercise 1.14. Give an example to show that Conjecture 1.1 fails in general if $A \to R$ is module-finite but R is not torsion-free as an A-module.

Exercise 1.15. Prove that in characteristic $p > 0$, Conjecture 1.1 holds as long as S splits from all its module-finite extensions (such S is called a splinter, we will discuss splinters in more details later).

Exercise 1.16. Prove that Conjecture 1.1 implies the following: Let $(R, \mathfrak{m})$ be a complete local domain and let P be a nonzero prime ideal of R such that R/P is a regular local ring of dimension d. Then the induced map $H_{\mathfrak{m}}^d(R) \to H_{\mathfrak{m}}^d(R/P)$ vanishes.

1.1. **Strong direct summand conjecture.** Work of Ranganathan [Ran00] shows that the vanishing conjecture for maps of Tor is equivalent to the following strong direct summand conjecture in all characteristics.

Conjecture 1.17. *Let $R \to S$ be a module-finite domain extension. Suppose R is regular local and x is part of a regular system of parameters of R (i.e., R and R/xR are both regular). Let Q be a height one prime ideal of S lying over xR. Then xR is a direct summand of Q considered as R-modules.*

We will not prove the equivalence here, the interested reader can refer to [Ran00, Chapter 5]. But we mention that this is indeed a huge generalization of the direct summand conjecture: since we have $xR \to xS \subseteq Q$, if $xR \to Q$ splits then so is $xR \to xS$, divide by x we see $R \to S$ splits.

1.2. **Recent progress on the homological conjectures.** Recently, the direct summand conjecture was proved by Yves André using perfectoid spaces [And16b], [And16a] (a simpler proof can be found in [Bha16]). It was also established in [And16a] that big Cohen-Macaulay algebra exists for complete local domains in mixed characteristics, and even more, for any injective homomorphism $R \to S$ between complete local domains, one can construct weakly functorial big Cohen-Macaulay algebras [And16a, Remarque 4.2.1].

2. Connections with splinters and derived splinters

Throughout this section, all rings are Noetherian and excellent, all schemes are Noetherian, excellent, and separated (in fact to avoid pathology, it is harmless to assume that all rings and schemes are essentially of finite type over a field or a complete DVR).

Definition 2.1. A domain S (resp., an integral scheme X) is called a *splinter*, if for every module-finite extension T of S (resp., every finite surjective map $Y \to X$), the natural map $S \to T$ (resp., $O_X \to O_Y$) is split in the category of S-modules (resp., O_X-modules).

Exercise 2.2. Prove that if S is a splinter, then S is normal. Moreover, prove that in characteristic 0, S is a splinter if and only if S is normal.

It turns out that even in characteristic $p > 0$, splinters are still mysterious.

Conjecture 2.3. *Let $(S, \mathfrak{n})$ be an excellent local domain of characteristic $p > 0$. Then S is a splinter if and only if S is strongly F-regular.*

It is well known that strong F-regularity (in fact even weak F-regularity) implies splinter [HH90].[1] The difficult part is the other direction. The best partial results of this conjecture can be found in [Sin99] and [CEMS14].

Below we sketch a proof that splinters in characteristic $p > 0$ are F-rational using Theorem 1.9, which already indicates the dramatic difference with the characteristic 0 case (for deeper results we refer to [Smi94]). Recall that an excellent local domain $(R, \mathfrak{m})$ of characteristic $p > 0$ is called F-rational if it is Cohen-Macaulay and $H^d_{\mathfrak{m}}(R)$ is simple in the category of modules with Frobenius action (i.e., there is no nontrivial proper Frobenius stable submodule of $H^d_{\mathfrak{m}}(R)$).

Exercise 2.4. Prove that if an excellent local domain $(R, \mathfrak{m})$ is F-rational on $\operatorname{Spec} R - \{\mathfrak{m}\}$, then the largest proper Frobenius stable submodule of $H^d_{\mathfrak{m}}(R)$ has finite length.

Theorem 2.5. *Let $(S, \mathfrak{n})$ be a local domain of characteristic $p > 0$ that is a homomorphic image of a Gorenstein local ring. If S is a splinter, then S is F-rational.*

Proof. We pick a counter-example $(S, \mathfrak{n})$ of smallest dimension. Since the splinter property localizes, we know that $(S, \mathfrak{n})$ is F-rational on the punctured spectrum $\operatorname{Spec} S - \{\mathfrak{n}\}$. Exercise 2.4 implies that the largest proper Frobenius stable submodule of $H^d_{\mathfrak{n}}(S)$ has finite length.

[1] As we will not use deep results in tight closure theory, we omit the precise definition of F-regularity. The interested reader should refer to [HH90].

Call this module N. By Theorem 1.9, N maps to 0 in $H^d_{\mathfrak{n}}(T)$ for some module-finite extension T of S. But since S is a splinter, $H^d_{\mathfrak{n}}(S) \to H^d_{\mathfrak{n}}(T)$ is injective. Hence $N = 0$ and thus $H^d_{\mathfrak{n}}(S)$ is simple as a module with Frobenius action.

It remains to show S is Cohen-Macaulay. We know $H^i_{\mathfrak{n}}(S) \to H^i_{\mathfrak{n}}(S^+)$ is injective because S is a splinter. But for $i < d$, $H^i_{\mathfrak{n}}(S^+) = 0$ since S^+ is balanced big Cohen-Macaulay by Theorem 1.8. This implies $H^i_{\mathfrak{n}}(S) = 0$ for $i < d$ and so S is Cohen-Macaulay. □

In mixed characteristic, some experiments in low dimension show that splinters behave more like characteristic $p > 0$ situation.

Exercise 2.6. Let k be a perfect field and a, b, c positive integers. Prove that $k[x, y, z]/(x^a + y^b + z^c)$ is always a splinter if k has characteristic 0. Prove that $k[x, y, z]/(x^a + y^b + z^c)$ is a splinter in characteristic $p \gg 0$ if and only if $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1$.

Exercise 2.7. Let $\mathbb{Z}_p$ be the p-adic integers and a, b, c positive integers. Prove that, when $p \gg 0$, $\mathbb{Z}_p[y, z]/(p^a + y^b + z^c)$ is a splinter if and only if $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1$.

To the best of the author's knowledge, the following question is open.

Question 2.8. Let $(R, \mathfrak{m})$ be a splinter in characteristic $p > 0$ or mixed characteristic. Is $R[x]$ or $R[[x]]$ still a splinter?

To discuss the connection between splinters and the vanishing conjecture for maps of Tor, we begin with the following definition from [Ma15].

Definition 2.9. We say a local domain $(S, \mathfrak{n})$ satisfies the vanishing conditions for maps of Tor, if for every $A \to R \to S$ such that A is a regular domain, $A \to R$ is a module-finite torsion-free extension, and A, R, S have the same characteristic, the natural map $\operatorname{Tor}_i^A(M, R) \to \operatorname{Tor}_i^A(M, S)$ vanishes for every A-module M and every $i \geq 1$.

We next establish the connection between splinters and rings that satisfy the vanishing conditions for maps of Tor [Ma15, Corollary 4.5]. We begin with the following lemma:

Lemma 2.10. *Let $A \to B$ be a module-finite extension. Suppose $Q \in \operatorname{Spec} B$ lies over $P \in \operatorname{Spec} A$. If $P \to Q$ splits as A-modules and $\operatorname{depth}_P A \geq 2$, then $A \to B$ splits compatibly with $P \to Q$, i.e., there exists a splitting θ: $B \to A$ such that $\theta(Q) = P$. In particular, $A/P \to B/Q$ splits as A-modules.*

Proof. Let ϕ: $Q \to P$ be a splitting. The exact sequences $0 \to Q \to B \to B/Q \to 0$ and $0 \to P \to A$ induce a commutative diagram:

$$\begin{tikzcd} \operatorname{Hom}_A(B, A) \arrow[r] & \operatorname{Hom}_A(Q, A) \arrow[r] & \operatorname{Ext}^1_A(B/Q, A) . \\ & \operatorname{Hom}_A(Q, P) \arrow[u, hook] & \end{tikzcd}$$

Since B/Q is a finitely generated A-module annihilated by P and $\operatorname{depth}_P A \geq 2$, we know that $\operatorname{Ext}^1_A(B/Q, A) = 0$. Hence $\operatorname{Hom}_A(B, A)$ maps onto $\operatorname{Hom}_A(Q, A)$, in particular it maps onto the image of $\operatorname{Hom}_A(Q, P)$. Thus there is a map θ: $B \to A$ such that $\theta|_Q = \phi$. We show that θ has to be a splitting from B to A. Suppose $\theta(1) = a \in A$, for every nonzero element $r \in P$, we have

$$ra = r\theta(1) = \theta(r) = \phi(r) = r.$$

So $a = 1$ and hence θ is a splitting from B to A such that $\theta(Q) = P$, i.e., θ compatibly splits $P \to Q$. Finally $\overline{\theta}$ gives a splitting from $B/Q \to A/P$. This finishes the proof. $\square$

Theorem 2.11. *Let S be a homomorphic image of a regular local ring. If S satisfies the vanishing conditions for maps of Tor, then S is a splinter.*

Proof. We write $S = A/P$ such that A is a regular local ring with $\operatorname{depth}_P A \geq 2$ (this can be achieved, for example, by adding indeterminants). Let $S \to T$ be a module-finite domain extension. Let $t_1, \dots, t_n$ be a set of generators of T over $S = A/P$. Each t_i satisfies a monic polynomial f_i over S. We lift each f_i to A and form the ring $B = \frac{A[x_1,\dots,x_n]}{(f_1,\dots,f_n)}$. We have a natural surjection $B \twoheadrightarrow T$ with kernel $Q \in \operatorname{Spec} B$. It is straightforward to check that Q lies over P.

We form the ring $R = A + Q \subseteq B$. Then R is also a module-finite torsion-free extension of A and we have $R/Q = A/P = S$. Now we look at the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & Q \arrow[r] & R \arrow[r] & S \arrow[r] & 0 . \\
0 \arrow[r] & P \arrow[r] \arrow[u, "\alpha"] & A \arrow[r] \arrow[u, "\beta"] & S \arrow[r] \arrow[u, "\cong"] & 0
\end{tikzcd}$$

Tensoring the above diagram with an arbitrary A-module M, we get:

$$\begin{tikzcd}
\operatorname{Tor}_1^A(M,R) \arrow[r, "\varphi_M"] & \operatorname{Tor}_1^A(M,S) \arrow[r] & Q \otimes_A M \arrow[r] & R \otimes_A M \arrow[r] & S \otimes_A M \arrow[r] & 0 . \\
0 \arrow[r] & \operatorname{Tor}_1^A(M,S) \arrow[r] \arrow[u, "\cong"] & P \otimes_A M \arrow[r] \arrow[u, "\alpha\otimes\mathrm{id}_M"] & A \otimes_A M \arrow[r] \arrow[u, "\beta\otimes\mathrm{id}_M"] & S \otimes_A M \arrow[r] \arrow[u, "\cong"] & 0
\end{tikzcd}$$

By a diagram chasing, one can see that

$$\alpha \otimes \mathrm{id}_M \text{ is injective} \iff \varphi_M = 0 \text{ and } \beta \otimes \mathrm{id}_M \text{ is injective.} \tag{2.11.1}$$

Since B is finite free over A, we know that $A \to B$ splits as a map of A-modules and thus $A \to R$ also splits. In particular $\beta \otimes \mathrm{id}_M$ is injective. But since S satisfies the vanishing conditions for maps of Tor, $\varphi_M = 0$ and hence (2.11.1) tells us $\alpha \otimes \mathrm{id}_M$ is injective for every M. But this implies $P \to Q$ splits by Corollary 5.2 in [HR76] since Q/P is a finitely generated A-module. Since $\operatorname{depth}_P A \geq 2$, by Lemma 2.10, $S = A/P \to B/Q = T$ splits as a map of A-modules (hence also as a map of S-modules). As this is true for any module-finite domain extension T of S, S is a splinter. $\square$

Corollary 2.12. *In characteristic $p > 0$, S satisfies the vanishing conditions for maps of Tor if and only if S is a splinter.*

Proof. This follows from Theorem 2.11 and Exercise 1.15. $\square$

We next discuss derived splinters.

Definition 2.13. An integral scheme X is called a *derived splinter*, if for any proper surjective map $f\colon Y \to X$, the pullback map $O_X \to \mathbf{R}f_*O_Y$ is split in the derived category $D(\mathrm{Coh}(X))$ of coherent sheaves on X. This is the same as requiring $O_X \to \mathbf{R}f_*O_Y$ to split in $D(\mathrm{QCoh}(X))$, the derived category of quasi-coherent sheaves on X.

Exercise 2.14. Prove that to show X is a derived splinter, it is enough to check $O_X \to \mathbf{R}f_*O_Y$ is split for all alterations $Y \to X$ (i.e., proper generically finite dominant maps with Y smooth).

Quite obviously, if X is a derived splinter, then it is a splinter. In equal characteristic, derived splinters are well understood:

Theorem 2.15. *(1) In characteristic* 0*, derived splinters are the same as rational singularities*[2] [Kov00]*,* [Bha12]*.*
(2) In characteristic $p > 0$*, derived splinters are the same as splinters* [Bha12].

The crucial ingredient in Bhatt's proof of the above theorem in characteristic $p > 0$ is a cohomology annihilating result, which can be viewed as a relative version of Theorem 1.9.

Theorem 2.16. *Let* $f\colon Y \to X$ *be a proper morphism of Noetherian schemes of characteristic* $p > 0$*. Then there exists a finite surjective morphism* $\pi\colon Z \to Y$ *such that the pullback map* $\pi^*\colon \tau_{\geq 1}\mathbf{R}f_*O_Y \to \tau_{\geq 1}\mathbf{R}f_*\mathbf{R}\pi_*O_Z$ *is the zero map.*

It is not hard to prove Theorem 2.15 in characteristic $p > 0$ given Theorem 2.16. An alternative proof of Theorem 2.15 in characteristic $p > 0$ using big Cohen-Macaulayness of R^+ can be found in [Ma15, Remark 5.13].

Proof of Theorem 2.15 (2). Let X be a splinter of characteristic $p > 0$. Let $f\colon Y \to X$ be a proper surjective morphism. By Theorem 2.16, there exists a finite surjective morphism π: $Z \to Y$ such that $\tau_{\geq 1}\mathbf{R}f_*O_Y \to \tau_{\geq 1}\mathbf{R}f_*\mathbf{R}\pi_*O_Z$ is the zero map. Consider the diagram

$$\begin{tikzcd}
f_*\pi_*O_Z \arrow[r] & \mathbf{R}f_*\mathbf{R}\pi_*O_Z \arrow[r] & \tau_{\geq 1}\mathbf{R}f_*\mathbf{R}\pi_*O_Z \arrow[r, "+1"] & {} \\
f_*O_Y \arrow[r] \arrow[u] & \mathbf{R}f_*O_Y \arrow[r] \arrow[u] & \tau_{\geq 1}\mathbf{R}f_*O_Y \arrow[r, "+1"] \arrow[u, "0"] & {}
\end{tikzcd}$$

Since the composite map $\mathbf{R}f_*O_Y \to \tau_{\geq 1}\mathbf{R}f_*\mathbf{R}\pi_*O_Z$ is 0, we have the following diagram of objects in the derived category:

$$\begin{tikzcd}
f_*\pi_*O_Z \arrow[r] & \mathbf{R}f_*\mathbf{R}\pi_*O_Z \arrow[r] & \tau_{\geq 1}\mathbf{R}f_*\mathbf{R}\pi_*O_Z \arrow[r, "+1"] & {} \\
\mathbf{R}f_*O_Y \arrow[r, "\mathrm{id}"] \arrow[u, dotted, "\alpha"] & \mathbf{R}f_*O_Y \arrow[r] \arrow[u] & 0 \arrow[r, "+1"] \arrow[u] & {}
\end{tikzcd}$$

Axioms of triangulated category imply the dotted map α exists: i.e., the map

$$\mathbf{R}f_*O_Y \to \mathbf{R}f_*\mathbf{R}\pi_*O_Z$$

factors through $f_*\pi_*O_Z$. Because $f_*\pi_*O_Z$ is a coherent sheaf of algebras of O_X, it corresponds to the structure sheaf of a finite surjective morphism. Hence $O_X \to f_*\pi_*O_Z$ admits a splitting β since X is a splinter. Now $\beta \circ \alpha$ gives a splitting $\mathbf{R}f_*O_Y \to O_X$. $\square$

Theorem 2.15 suggests the following question:

Question 2.17. Are splinters and derived splinters the same in mixed characteristic?

The above question is wide open. A weaker question is the following:

[2] X has rational singularities in characteristic 0 if for a resolution of singularities $\pi\colon Y \to X$, $\mathbf{R}\pi_*O_Y = O_X$.

Question 2.18. Let $(R, \mathfrak{m})$ be a splinter in mixed characteristic. Is it true that $R[1/p]$ has rational singularities?

Exercise 2.19. Prove that Question 2.17 implies Question 2.18.

Let us mention that recently Bhargav Bhatt proved the derived direct summand conjecture in [Bha16]:

Theorem 2.20. *Let $(S, \mathfrak{n})$ be a regular local ring. Then S is a derived splinter.*

Below we give a proof of Theorem 2.20 in dimension two. This is different than Bhatt's approach, and is known to experts. First of all, the dimension one case is left as an exercise:

Exercise 2.21. Show that if $\dim S = 1$, then S is a derived splinter if and only if S is a splinter, and also if and only if S is regular.

Theorem 2.22. *Let $(S, \mathfrak{n})$ be a regular local ring of dimension two. Then S is a derived splinter.*

Proof. Pick $f \in S$ part of a regular system of parameters, i.e., S/fS is a regular local ring of dimension one. By Exercise 2.14, it is enough to check $S \to \mathbf{R}\pi_* O_Y$ is split for $Y \to \operatorname{Spec} S$ an alteration. Let E be the subscheme of Y defined by f. We have the following commutative diagram:

$$\begin{tikzcd}
S \arrow[r, "\times f"] \arrow[d, "\alpha"] & S \arrow[r] \arrow[d, "\alpha"] & S/fS \arrow[r, "+1"] \arrow[d, "\beta"] & {} \\
\mathbf{R}\pi_* O_Y \arrow[r, "\times f"] & \mathbf{R}\pi_* O_Y \arrow[r] & \mathbf{R}\pi_* O_E \arrow[r, "+1"] & {}
\end{tikzcd}$$

Applying $\mathbf{R}\operatorname{Hom}(-, \omega_S)$, by duality we have

$$\mathbf{R}\operatorname{Hom}(\mathbf{R}\pi_* O_Y, \omega_S) \cong \mathbf{R}\pi_*\omega_Y \cong \pi_*\omega_Y$$

where the last isomorphism is because Y is smooth of dimension two and we know $\mathbf{R}^1\pi_*\omega_Y$ vanishes since the Grauert-Riemenschneider vanishing holds in dimension two in all characteristics [Lip78]. So the dual of the above commutative diagram becomes:

$$\begin{tikzcd}
0 & \omega_{S/fS} \arrow[l] & \omega_S \arrow[l] & \omega_S \arrow[l, "\times f"'] & 0 \arrow[l] \\
0 & \pi_*\omega_E \arrow[l] \arrow[u, "\beta^\vee"] & \pi_*\omega_Y \arrow[l] \arrow[u, "\alpha^\vee"] & \pi_*\omega_Y \arrow[l, "\times f"'] \arrow[u, "\alpha^\vee"] & 0 \arrow[l]
\end{tikzcd}$$

Since S/fS is a regular local ring of dimension one and thus a derived splinter by Exercise 2.21, we know β is a split injection and hence $\beta^\vee$ is a split surjection. Chasing the diagram and using Nakayama's lemma shows that $\alpha^\vee$ is surjective. However, since S is regular, ω_S is free and thus $\alpha^\vee$ is a split surjection. Therefore α is a split injection. $\square$

Remark 2.23. The proof above does not generalize to higher dimension because we no longer have Grauert-Riemenschneider vanishing. In fact, it is known that Question 2.17 has a positive answer in dimension two [BM16], i.e., two dimensional splinters in mixed characteristic are derived splinters.

The main result in [Ma15] is the following, which largely generalizes Conjecture 1.1 in characteristic 0.

Theorem 2.24. *In characteristic 0, S satisfies the vanishing conditions for maps of Tor if and only if S has rational singularities.*

Corollary 2.12, Theorem 2.24 and Theorem 2.15 together tell us that, in equal characteristic, rings that satisfy the vanishing conditions for maps of Tor are exactly derived splinters. It is not known whether this is true in mixed characteristic:

Conjecture 2.25. *In mixed characteristic, S satisfies the vanishing conditions for maps of Tor if and only if S is a derived splinter.*

3. Appendix 1: Solution to Exercise 2.7

We split the solution of Exercise 2.7 into two separate theorems.[3]

Theorem 3.1. *For every (V, pV) complete unramified DVR, $R = \frac{V[[y,z]]}{p^a+y^b+z^c}$ is not a splinter when $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \le 1$.*

Proof. First we claim that, under the hypothesis,

$$y^{(b-1)p} \in (p^p, z^p, y^{b-1}p^{p-1}z^{p-1}). \tag{3.1.1}$$

To see this, note that we have

$$y^{(b-1)p} = y^\beta (y^b)^{\lfloor \frac{(b-1)p}{b} \rfloor} = y^\beta(-p^a - z^c)^{\lfloor \frac{(b-1)p}{b} \rfloor}$$

where $\beta = (b-1)p - b\lfloor \frac{(b-1)p}{b} \rfloor$. Expanding $y^\beta(-p^a - z^c)^{\lfloor \frac{(b-1)p}{b} \rfloor}$, we see that if a term $y^\beta p^{a\alpha} z^{c\gamma}$ is not in (p^p, z^p), then we must have $a\alpha \le p-1$, $c\gamma \le p-1$. So we have:

$$\lfloor \frac{(b-1)p}{b} \rfloor = \alpha + \gamma \le \frac{p-1}{a} + \frac{p-1}{c}.$$

Notice that (*) $\lfloor \frac{(b-1)p}{b} \rfloor = p - \lceil \frac{p}{b} \rceil \ge p - (\frac{p-1}{b} + 1)$, combining this with the above we get:

$$p - (\frac{p-1}{b} + 1) \le \frac{p-1}{a} + \frac{p-1}{c}.$$

Rewrite this we get

$$p - 1 \le \frac{p-1}{a} + \frac{p-1}{b} + \frac{p-1}{c} \le p - 1.$$

This means, if $y^\beta p^{a\alpha} z^{c\gamma} \notin (p^p, z^p)$, we must have $a\alpha = p-1$, $c\gamma = p-1$ and an "=" in (*), i.e., $b \mid p-1$. Now it is easy to check that when $b \mid p-1$, $\beta = (b-1)p - b\lfloor \frac{(b-1)p}{b} \rfloor = b-1$. Hence we have $y^\beta p^{a\alpha} z^{c\gamma} = y^{b-1}p^{p-1}z^{p-1}$. This finishes the proof of the claim.

Now we prove that R is not a splinter. It suffices to show $y^{b-1} \in (p, z)^+$, i.e., we can write $y^{b-1} = pv - zu$ for some u, v integral over R. We write

$$v = \frac{zu + y^{b-1}}{p}$$

and try to write down explicit equations that u, v satisfy. First of all we have

$$v^p = \frac{1}{p^p}(z^p u^p + \sum_{i=1}^{p-1} \binom{p}{i} y^{(b-1)i} z^{p-i} u^{p-i} + y^{(b-1)p}).$$

[3]These computations are well known at least to Anurag Singh.

I claim that we can rewrite the above equation in the following form:

$$v^p = \frac{1}{p^p}(z^p u^p + Ly^{b-1}p^{p-1}z^{p-1} + p^p M + z^p N_0 + \sum_{\Lambda} c^0_{\epsilon rst} y^\epsilon p^r z^s u^t) \tag{3.1.2}$$

where L and $c^0_{\epsilon rst}$ are integers, M is a polynomial in y, z, N_0 is a polynomial in y, z, u with the degree on u less than p, and Λ denotes the following:

$$\Lambda = \begin{cases} 0 \le \epsilon \le b-1 \\ 0 \le t \le p-1 \\ t \le s \le p-1 \\ \epsilon ac + rbc + sab + t(abc - ac - ab) = p(abc - ac) + bc \end{cases}$$

This is because by (3.1.1), we can write $y^{(b-1)p} = Ly^{b-1}p^{p-1}z^{p-1} + p^p M + z^p N'$ for L some integer, and M, N' polynomials in y, z. For the other terms $\binom{p}{i} y^{(b-1)i} z^{p-i} u^{p-i}$, we can keep factoring $y^b = -p^a - z^c$ from each $y^{(b-1)i}$ until the exponent of y is less than b, then we expand it in terms of p and z and we get an expression which either is divisible by z^p, or satisfies Λ. Note that the last condition in Λ is easily checked if we think of

$$\deg y = ac, \deg p = bc, \deg z = ab, \deg u = abc - ac - ab$$

because under this "grading", $zu + y^{b-1}$ and $p^a + y^b + z^c$ are both "homogeneous". We can combine those terms which are divisible by z^p with $z^p N'$ (coming from $y^{(b-1)p}$) to get $z^p N_0$. We have verified (3.1.2).

We next introduce the following conditions Λ_k:

$$\Lambda_k = \begin{cases} 0 \le \epsilon \le b-1 \\ 0 \le t \le p-k-1 \\ t \le s \le p-1 \\ \epsilon ac + rbc + sab + t(abc - ac - ab) = p(abc - ac) + bc \end{cases}$$

It is ready to see that $\Lambda = \Lambda_0$.

I claim that for every $0 \le k \le p-1$, there exist Q_i, polynomials in y, z, such that

$$v^p - \sum_{i=1}^{k} Q_i v^{p-i} = \frac{1}{p^p}(z^p u^p + Ly^{b-1}p^{p-1}z^{p-1} + p^p M + z^p N_k + \sum_{\Lambda_k} c^k_{\epsilon rst} y^\epsilon p^r z^s u^t) \tag{3.1.3}$$

where $c^k_{\epsilon rst}$ are integers, N_k are polynomials in y, z, u with the degree on u less than p.

We prove this by induction on k. When $k = 0$ this is exactly (3.1.2). Suppose $k \ge 1$ and we have this expression for $k-1$, that is, we have

$$v^p - \sum_{i=1}^{k-1} Q_i v^{p-i} = \frac{1}{p^p}(z^p u^p + Ly^{b-1}p^{p-1}z^{p-1} + p^p M + z^p N_{k-1} + \sum_{\Lambda_{k-1}} c^{k-1}_{\epsilon rst} y^\epsilon p^r z^s u^t) \tag{3.1.4}$$

Notice that in $\sum_{\Lambda_{k-1}} c^{k-1}_{\epsilon rst} y^\epsilon p^r z^s u^t$, the highest u terms are $c^{k-1}_{\epsilon rs(k-1)} y^\epsilon p^r z^s u^{p-k}$ with $s \ge p-k$, because the condition Λ_{k-1} implies $s \ge t$ and $t \le p-k$. For each such highest u term $c^{k-1}_{\epsilon rs(k-1)} y^\epsilon p^r z^s u^{p-k}$, I claim that $r \ge k$. Because otherwise,

$$\epsilon \le b-1, r \le k-1, s \le p-1, t = p-k,$$

and we will have

$$\begin{array}{rl} & \epsilon ac + rbc + sab + t(abc - ac - ab) \\ \le & (b-1)ac + (k-1)bc + (p-1)ab + (p-k)(abc - ac - ab) \\ = & (p-k+1)(abc - ac) + (k-1)(ab + bc) \\ < & p(abc - ac) + bc \end{array}$$

which contradicts the last condition in Λ_{k-1} (the last $<$ in the above inequalities is obvious if $k = 1$, and is a consequence of $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \le 1$ when $k > 1$).

For each term $c^{k-1}_{\epsilon rs(k-1)} y^\epsilon p^r z^s u^{p-k}$, we assign $\Delta_{\epsilon rs} = r - k \ge 0$. We let

$$Q_k = \sum_{0 \le \epsilon \le b, p-k \le s \le p-1} c^{k-1}_{\epsilon rs(k-1)} y^\epsilon p^{\Delta_{\epsilon rs}} z^{s-(p-k)}.$$

Now we compute $Q_k v^{p-k}$. First we recall that v^{p-k}:

$$\begin{aligned} v^{p-k} &= \frac{1}{p^{p-k}}(z^{p-k}u^{p-k} + \sum_{j=1}^{p-k} \binom{p-k}{j} y^{2j} z^{p-k-j} u^{p-k-j}) \\ &= \frac{1}{p^p}(p^k z^{p-k} u^{p-k} + \sum_{j=1}^{p-k} p^k \binom{p-k}{j} y^{2j} z^{p-k-j} u^{p-k-j}). \end{aligned}$$

I claim that we can write $Q_k v^{p-k}$ in the following form:

$$Q_k v^{p-k} = \frac{1}{p^p}(\sum_{\Lambda_{k-1}, t=p-k} c^{k-1}_{\epsilon rs(p-k)} y^\epsilon p^r z^s u^{p-k} + \sum_{\Lambda_k} \tilde{c}^k_{\epsilon rst} y^\epsilon p^r z^s u^t + z^p \tilde{N}_k) \tag{3.1.5}$$

where $\tilde{c}^k_{\epsilon rst}$ are integers and $\tilde{N}_k$ is a polynomial in y, z, u with degree on u less than p. To see this, note that multiplying Q_k by $p^k z^{p-k} u^{p-k}$ will give us the first term

$$\sum_{\Lambda_{k-1}, t=p-k} c^{k-1}_{\epsilon rs(p-k)} y^\epsilon p^r z^s u^{p-k}$$

in (3.1.5). While when we multiply $c^{k-1}_{\epsilon rs(k-1)} y^\epsilon p^{\Delta_{\epsilon rs}} z^{s-(p-k)}$ in Q_k with $p^k \binom{p-k}{j} y^{2j} z^{p-k-j} u^{p-k-j}$ in v^{p-k}, we get

$$c^{k-1}_{\epsilon rs(k-1)} \binom{p-k}{j} y^{\epsilon+2j} p^{\Delta_{\epsilon rs}+k} z^{s-(p-k)+p-k-j} u^{p-k-j}$$

we can keep factoring $y^b = -p^a - z^c$ from $y^{2j+\epsilon}$ until the exponent of y is less than b, and when we expand, each term would either be divisible by z^p, or satisfy condition Λ_k, because the exponent of u never gets larger than $p-k-1$, the exponent of z can only be greater than the exponent of u (since we start with $z^{s-(p-k)+p-k-j} u^{p-k-j}$), and the last equation in Λ_k is satisfied because of "homogeneous" reasons.

The key point is that in (3.1.5), the first term is the same as the highest term of $\sum_{\Lambda_{k-1}} c_{\epsilon rst}^{k-1} y^\epsilon p^r z^s u^t$ in (3.1.4), i.e, those terms with $t = p-k$. Therefore we have

$$\begin{aligned} v^p - \sum_{i=1}^{k} Q_i v^{p-i} &= \frac{1}{p^p}(z^p u^p + L y^{b-1} p^{p-1} z^{p-1} + p^p M + z^p (N_{k-1} - \widetilde{N}_k) + \sum_{\Lambda_k} (c_{\epsilon rst}^{k-1} - \widetilde{c}_{\epsilon rst}^{k}) y^\epsilon p^r z^s u^t) \\ &:= \frac{1}{p^p}(z^p u^p + L y^{b-1} p^{p-1} z^{p-1} + p^p M + z^p N_k + \sum_{\Lambda_k} c_{\epsilon rst}^{k} y^\epsilon p^r z^s u^t) \end{aligned}$$

where $c_{\epsilon rst}^{k} = c_{\epsilon rst}^{k-1} - \widetilde{c}_{\epsilon rst}^{k}$ are integers, and $N_k = N_{k-1} - \widetilde{N}_k$ is a polynomial in y, z, u with degree on u less than p. This finishes the induction step, and hence the proof of (3.1.3).

Now we apply (3.1.3) with $k = p-1$, as the condition Λ_{p-1} implies $t = 0$, we get

$$v^p - \sum_{i=1}^{p-1} Q_i v^{p-i} = \frac{1}{p^p}(z^p u^p + L y^{b-1} p^{p-1} z^{p-1} + p^p M + z^p N_{p-1} + \sum_{\Lambda_{p-1}} c_{\epsilon rs0}^{p-1} y^\epsilon p^r z^s). \tag{3.1.6}$$

Each $y^\epsilon p^r z^s$ inside (3.1.6) satisfies $\epsilon \le b-1$ and $s \le p-1$, so by the Λ_{p-1} condition, we know $rbc \ge p(abc - ac) + bc - (p-1)ab - (b-1)ac$. Dividing by abc, we get

$$\frac{r-1}{a} \ge p - 1 - \frac{p-1}{c} - \frac{p-1}{b}$$

from which we know $r \ge p$ because $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \le 1$.

Therefore, we can rewrite (3.1.6) as

$$\begin{aligned} v^p - \sum_{i=1}^{p-1} Q_i v^{p-i} &= \frac{1}{p^p}(z^p u^p + L y^{b-1} p^{p-1} z^{p-1} + p^p \widetilde{M} + z^p N_{p-1}) \\ &= \widetilde{M} + \frac{L y^{b-1} z^{p-1}}{p} + \frac{z^p (u^p + N_{p-1})}{p^p} \end{aligned} \tag{3.1.7}$$

where $\widetilde{M}$ is a polynomial in y, z and N_{p-1} is a polynomial in y, z, u with degree on u less than p.

At this point, recall that $v = \frac{zu + y^{b-1}}{p}$, so we have

$$\frac{L y^{b-1} z^{p-1}}{p} = L z^{p-1}(v - \frac{zu}{p}) = L z^{p-1} v - \frac{z^p L p^{p-1} u}{p^p}.$$

Plugging in this into (3.1.7), we get

$$\begin{aligned} v^p - \sum_{i=1}^{p-1} Q_i v^{p-i} - L z^{p-1} v &= \widetilde{M} + \frac{z^p (u^p + N_{p-1} - L p^{p-1} u)}{p^p} \\ &:= \widetilde{M} + \frac{z^p (u^p + \widetilde{N})}{p^p} \end{aligned} \tag{3.1.8}$$

where $\widetilde{N} = N_{p-1} - L p^{p-1} u$ is a polynomial in y, z, u with degree on u less than p.

Finally, let u be the solution of the monic polynomial (remember the degree of u in $\widetilde{N}$ is less than p):

$$u^p + \widetilde{N} = 0$$

setting $v = \frac{zu+y^{b-1}}{p}$, we know from (3.1.8) that v satisfies

$$v^p - \sum_{i=1}^{p-1} Q_i v^{p-i} - Lz^{p-1}v = \widetilde{M}$$

which is a monic polynomial of v with coefficients in R (remember Q_i, $\widetilde{M}$ are polynomials in y, z). Therefore, we can let u, v be the solutions of the system

$$\begin{cases} u^p + \widetilde{N} = 0 \\ v^p - \sum_{i=1}^{p-1} Q_i v^{p-i} - Lz^{p-1}v = \widetilde{M}. \end{cases}$$

Then u, v are both integral over R and $y^{b-1} = pv - zu$ hence $y^{b-1} \in (p, z)^+$ as desired. $\square$

Remark 3.2. Another way to prove Theorem 3.1 is to use the fact that two dimensional splinters in mixed characteristic are derived splinters [BM16], in particular they are pseudo-rational. But it is easy to prove that $R = \frac{V[[y,z]]}{p^a+y^b+z^c}$ is not pseudo-rational when $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq 1$, using an explicit computation by desingularization.

We next prove the other direction in Exercise 2.7. This follows from the following theorem (we use $W(\overline{\mathbb{F}_p})$ to denote the ring of Witt vectors over $\overline{\mathbb{F}_p}$).

Theorem 3.3. *If $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1$, then for $p \gg 0$,*

$$R = \frac{W(\overline{\mathbb{F}_p})[[y, z]]}{p^a + y^b + z^c}$$

is a direct summand of some regular local ring. Hence it is a splinter for $p \gg 0$.

Proof. By the classical theory of Du Val singularities [Rei], $\frac{\mathbb{C}[x,y,z]}{x^a+y^b+z^c}$ is isomorphic to the ring of invariants of $\mathbb{C}[u, v]$ under a certain finite group action when $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1$. In particular, we know that there is a map

$$\frac{\mathbb{C}[x, y, z]}{x^a + y^b + z^c} \to \mathbb{C}[u, v]$$

where $x \to f(u, v)$, $y \to g(u, v)$, $z \to h(u, v)$, such that $\mathbb{C}[u, v]$ is module-finite over the base. Since f, g, h have only finitely many coefficients, we know that there exists a number field $\mathbb{Q}(\lambda)$ such that $f, g, h \in \mathbb{Q}(\lambda)[u, v]$. Since the coefficients of f, g, h involve only finitely many denominators, we know for $p \gg 0$, $f, g, h \in \mathbb{Z}_{(p)}(\lambda)[u, v]$, and the relation $f^a + g^b + h^c = 0$ still holds in $\mathbb{Z}_{(p)}(\lambda)[u, v]$. As $\mathbb{Z}_{(p)}(\lambda) \subseteq W(\overline{\mathbb{F}_p})$, we know that for $p \gg 0$, there are elements $f, g, h \in W(\overline{\mathbb{F}_p})[u, v]$ such that $f^a + g^b + h^c = 0$.

Now we define a map

$$R = \frac{W(\overline{\mathbb{F}_p})[[y, z]]}{p^a + y^b + z^c} \to S = \frac{W(\overline{\mathbb{F}_p})[[u, v]]}{p - f(u, v)}$$

where $y \to g(u, v)$, $z \to h(u, v)$. It is readily seen that this map is well-defined, because $f^a + g^b + h^c = 0$ in $W(\overline{\mathbb{F}_p})[u, v]$. It is straightforward to check that $R \to S$ is a module-finite extension with S a (ramified) regular local ring of dimension 2. In fact, one can check that

$$\operatorname{rank}_R S = \operatorname{rank}_{\frac{\mathbb{C}[x,y,z]}{x^a+y^b+z^c}} \mathbb{C}[u, v].$$

From the construction it is obvious that the latter one is the same as $\operatorname{rank}_{\frac{W(\overline{\mathbb{F}_p})[x,y,z]}{x^a+y^b+z^c}} W(\overline{\mathbb{F}_p})[u,v]$ where x, y, z are mapped to f, g, h (and killing $p - x$ does not change the rank). Therefore for $p \gg 0$, the trace map will give us a splitting from S to R. $\square$

4. Appendix 2: Further observations on Conjecture 1.1

Recently, Hochster proposed the following new conjecture [Hoc16, Conjecture 14.5]:

Conjecture 4.1 (Next to top local cohomology is almost zero)**.** *Let $(R, \mathfrak{m})$ be a complete local domain of dimension d of mixed characteristic. Then $H^{d-1}_{\mathfrak{m}}(R^+)$ is killed by $\mathfrak{m}_{R^+}$.*

It was pointed out in [Hoc16] that this conjecture implies the direct summand conjecture. We want to observe here that, with the recent solution of the direct summand conjecture [And16a] [Bha16], the above conjecture implies the vanishing conjecture for maps of Tor.

Theorem 4.2. *Conjecture 4.1 implies Conjecture 1.1.*

To establish this we need a couple lemmas. The first one was taken from Hochster-Huneke [HH95], and Ranganathan [Ran00].

Lemma 4.3. *In any characteristic, to prove Conjecture 1.1, it suffices to prove it for $i = 1$, and we can assume $(A, \mathfrak{m})$ is complete local, R is a complete local domain, and $S = A/x$ where $x \in \mathfrak{m} - \mathfrak{m}^2$.*

The second lemma reduces Conjecture 1.1 to a vanishing on local cohomology.

Lemma 4.4. *To prove Conjecture 1.1, it suffices to show $\operatorname{Tor}_1^A(E, R) \to \operatorname{Tor}_1^A(E, S)$ is the zero map where E is the injective hull of the residue field of A. That is, if $\dim A = \dim R = d$, it is enough to prove $H^{d-1}_{\mathfrak{m}}(R) \to H^{d-1}_{\mathfrak{m}}(S)$ is the zero map.*

Proof. By Lemma 4.3, we may assume $(A, \mathfrak{m})$ is complete local and $S = A/x$. Let $A \to R \to S$ be as in Conjecture 1.1. Let Q be the kernel of $R \twoheadrightarrow S$. We have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & Q \arrow[r] & R \arrow[r] & S \arrow[r] & 0 . \\
0 \arrow[r] & (x) \arrow[r] \arrow[u, "\alpha"] & A \arrow[r] \arrow[u, "\beta"] & S \arrow[r] \arrow[u, "\cong"] & 0
\end{tikzcd}$$

Tensoring the above diagram with an arbitrary A-module M, we get:

$$\begin{tikzcd}
\operatorname{Tor}_1^A(M,R) \arrow[r, "\varphi_M"] & \operatorname{Tor}_1^A(M,S) \arrow[r] & Q \otimes_A M \arrow[r] & R \otimes_A M \arrow[r] & S \otimes_A M \arrow[r] & 0 . \\
0 \arrow[r] & \operatorname{Tor}_1^A(M,S) \arrow[r] \arrow[u, "\cong"] & (x) \otimes_A M \arrow[r] \arrow[u, "\alpha\otimes \mathrm{id}_M"] & A \otimes_A M \arrow[r] \arrow[u, "\beta\otimes \mathrm{id}_M"] & S \otimes_A M \arrow[r] \arrow[u, "\cong"] & 0
\end{tikzcd}$$

Since A is regular, by the direct summand conjecture (which holds in all characteristics now thanks to [And16a] [Bha16]), β is split and thus $\beta \otimes \mathrm{id}_M$ is always an injection. By a diagram chasing (which is entirely similar as in the proof of Theorem 2.11), we have

$$\alpha \otimes \mathrm{id}_M \text{ is injective} \Longleftrightarrow \varphi_M = 0. \tag{4.4.1}$$

Now suppose $\operatorname{Tor}_1^A(E, R) \to \operatorname{Tor}_1^A(E, S)$ is the zero map, that is, $\varphi_E = 0$. By (4.4.1), this implies $\alpha \otimes \mathrm{id}_E$ is injective. Hence the map $(x) \to Q$ is split because (x) is isomorphic to A

as an A-module and A is complete. But then we have $\alpha\otimes\mathrm{id}_M$ is injective for every M, thus by (4.4.1) again, $\varphi_M=0$ for every A-module M. Therefore Conjecture 1.1 holds for $i=1$. Lemma 4.3 thus implies Conjecture 1.1 holds in general. The last assertion of the lemma follows because the map $\mathrm{Tor}_1^A(E,R)\to\mathrm{Tor}_1^A(E,S)$ can be identified with the natural map $H^{d-1}_{\mathfrak{m}}(R)\to H^{d-1}_{\mathfrak{m}}(S)$. ☐

Proof of Theorem 4.2. By Lemma 4.4, to prove Conjecture 1.1, it is enough to show that for every $R\twoheadrightarrow S$ with R complete local domain and $S=R/P$ is complete regular local, where $\dim R=d$ and $\dim S=d-1$, the map $H^{d-1}_{\mathfrak{m}}(R)\to H^{d-1}_{\mathfrak{m}}(S)$ vanishes. Now we look at the following commutative diagram

$$\begin{tikzcd} H^{d-1}_{\mathfrak{m}}(R^+) \arrow[r] & H^{d-1}_{\mathfrak{m}}(S^+) \\ H^{d-1}_{\mathfrak{m}}(R) \arrow[r] \arrow[u] & H^{d-1}_{\mathfrak{m}}(S) \arrow[u] \end{tikzcd}$$

Assuming Conjecture 4.1, the image of $H^{d-1}_{\mathfrak{m}}(R)$ in $H^{d-1}_{\mathfrak{m}}(S^+)$ is killed by $\mathfrak{m}_{S^+}$. We pick a regular system of parameters $x_1,\dots,x_{d-1}$ of S. If $H^{d-1}_{\mathfrak{m}}(R)\to H^{d-1}_{\mathfrak{m}}(S)$ does not vanish, then its image must contain the socle $[\frac{1}{x_1\cdots x_{d-1}}]$. Therefore it is enough to show that the image of $[\frac{1}{x_1\cdots x_{d-1}}]$ in $H^{d-1}_{\mathfrak{m}}(S^+)$ is not killed by $\mathfrak{m}_{S^+}$. This follows formally from the direct summand conjecture and we give an argument below.

If S has equal characteristic $p>0$, then $S\cong K[[x_1,\dots,x_{d-1}]]$. In this case clearly $[\frac{1}{x_1\cdots x_{d-1}}]$ in $H^{d-1}_{\mathfrak{m}}(S^+)$ is not killed by $\mathfrak{m}_{S^+}$ (basically because 1 is not in the tight closure of $(x_1,\dots,x_{d-1})$). If S has mixed characteristic, then we can pick $S'\hookrightarrow S$ a module-finite extension such that $S'=V[[y_2,\dots,y_{d-1}]]$ for (V,pV) a complete unramified DVR. Since $[\frac{1}{py_2\cdots y_{d-1}}]$ is a multiple of $[\frac{1}{x_1\cdots x_{d-1}}]$, it is enough to show that the image of $[\frac{1}{py_2\cdots y_{d-1}}]$ in $H^{d-1}_{\mathfrak{m}}(S^+)$ is not killed by $\mathfrak{m}_{S^+}$. But we have $S'\hookrightarrow S''=V[[y_2^{1/p},\dots,y_{d-1}^{1/p}]]$. Since S'' is still a complete regular local ring, the direct summand conjecture ([And16a] or [Bha16]) tells us $S''\to S^+$ splits and thus $[\frac{1}{py_2^{1/p}\cdots y_{d-1}^{1/p}}]$ is not zero in $H^{d-1}_{\mathfrak{m}}(S^+)$. This implies $[\frac{1}{py_2\cdots y_{d-1}}]$ is not killed by $y_2^{(p-1)/p}\cdots y_{d-1}^{(p-1)/p}$. In particular it is not killed by $\mathfrak{m}_{S^+}$. ☐

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