Hypergeometric systems III: GKZ meets local cohomology

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Local cohomology RTG
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Outline

Recall: 1

- $A \in \mathbb{Z}^{d \times n}$, $\mathbb{Z}A = \mathbb{Z}^d$, pointed.
- $R_A = \mathbb{C}[\partial]$, $\mathcal{O}_A = \mathbb{C}[x]$, $D_A = \mathcal{O}_A\langle\partial\rangle$.
- $I_A = \ker(x_j \to t^{\mathbf{a}_j})$, $R_A/I_A = \mathbb{C}[\mathbb{N}A]$.
- $H_A(\beta) = D_A(I_A, E - \beta)$.

Euler–Koszul technology 2

Exercise: (with $E_i = a_{i,1}x_1\partial_1 + \ldots + a_{i,n}x_n\partial_n$)

$$\begin{aligned} x^{\mathbf{u}}E_i - E_i x^{\mathbf{u}} &= -(A\cdot\mathbf{u})_i x^{\mathbf{u}}, \\ \partial^{\mathbf{u}}E_i - E_i\partial^{\mathbf{u}} &= (A\cdot\mathbf{u})_i\partial^{\mathbf{u}}. \end{aligned}$$

- define $-\deg_A(x_j) = \mathbf{a}_j = \deg_A(\partial_j)$.
- E_i induces D_A-linear endo of $D_A \otimes_{R_A} S_A = D_A/I_A$ by

  $$E_i \circ (P \otimes 1) := PE_i \otimes 1 = (E_i + \deg_i(P))P \otimes 1$$

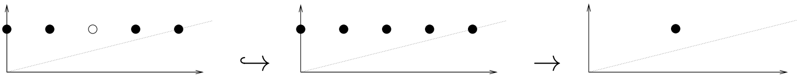
  where $\deg_A = (\deg_1, \ldots, \deg_d)$.
- $[E_i, E_j] = 0$; can form Koszul complex $\mathcal{K}_\bullet(S_A;\beta)$ on S_A.
- $\rightsquigarrow$ *Euler–Koszul functor*: $\mathbb{Z}^d$-graded R_A-mods $\to$ D_A-mods.
- Key detail: $\mathcal{H}_0(S_A;\beta) = M_A(\beta)$.

Rank versus volume: reminders 3

- A generic weight L gives a regular trinagulation of A
- To each simplex τ in the triangulation belong $\mathrm{vol}(\tau)$ many $\phi_{\mathbf{v}}$ where $A\mathbf{v} = \beta$.
- If β generic, all such $\phi_{\mathbf{v}}$ form basis of solutions for $M_A(\beta)$.

Rank $>$ vol I: the setup

$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}$. Will show: sometimes vol $<$ rk.



S_A $\qquad$ $\tilde{S}_A$ $\qquad$ $C \cong R_A/\partial\left(\begin{pmatrix} -1 \\ -2 \end{pmatrix}\right)$

- Each $N \in \{S_A, \tilde{S}_A, C\}$ gives Euler–Koszul complex

$$\begin{tikzcd} & \mathbb{C}[x] \otimes N \arrow[dr, "E_2 - \beta_2"] & \\ \mathbb{C}[x] \otimes N \arrow[ur, "E_1 - \beta_1"] \arrow[dr, "E_2 - \beta_2"'] & \oplus & \mathbb{C}[x] \otimes N \\ & \mathbb{C}[x] \otimes N \arrow[ur, "-E_1 + \beta_1"'] & \end{tikzcd}$$

Rank > vol II: the graded complex

- grade D_A by $x \mapsto 1$, $\partial \mapsto 0$. Note: $\mathrm{gr}(D_A) = \mathbb{C}[x, \partial]$.

$$\mathrm{gr} = \quad \begin{tikzcd} & \mathbb{C}[x] \otimes N \arrow[dr, "E_2"] & \\ \mathbb{C}[x] \otimes N \arrow[ur, "E_1"] \arrow[dr, "E_2"'] \arrow[rr, phantom, "\bigoplus"] & & \mathbb{C}[x] \otimes N \\ & \mathbb{C}[x] \otimes N \arrow[ur, "-E_1"'] & \end{tikzcd}$$

- If $N = \tilde{S}_A$, graded complex exact ($\tilde{S}_A$ is Cohen–Macaulay).
- Spectral sequence theorem:

$$\textit{graded complex exact} \Longrightarrow \textit{actual complex exact.}$$

Rank $>$ vol III: hunting for jumps

- L.e.s. of Euler–Koszul homology:

$$
\begin{tikzcd}
 & \mathcal{H}_0(S_A;\beta) \arrow[r] & \mathcal{H}_0(\tilde{S}_A;\beta) \arrow[r] & \mathcal{H}_0(C;\beta) \arrow[r] & 0 \\
 & \mathcal{H}_1(S_A;\beta) \arrow[r] & 0 \arrow[r] & \mathcal{H}_1(C;\beta) \arrow[ull] & \\
0 \arrow[r] & \mathcal{H}_2(S_A;\beta) \arrow[r] & 0 \arrow[r] & \mathcal{H}_2(C;\beta) \arrow[ull] &
\end{tikzcd}
$$

- Rank is additive... $\mathrm{rk}(\mathcal{H}_0(\tilde{S}_A,\beta))$ constant$=\mathrm{vol}(A)$...
- ...investigate $\mathcal{K}_\bullet(C;\beta)$!

Rank > vol IV: getting closer 7

What is $\mathcal{K}_\bullet(C;\beta)$ like?

- Recall: C sits in degree $(1,2)$.
- $\mathbb{C}[x]\otimes C = D_A/D_A(\partial_1,\dots,\partial_n)\cong \mathbb{C}[x]$ shifted by $(1,2)$:
- shift affects Euler–Koszul:
  $$E_1\circ P = (E_1+\deg_A(P)\boxed{+1})P = P(E_1\boxed{+1}) = P,$$
  $$E_2\circ P = (E_2+\deg_A(P)\boxed{+2})P = P(E_2\boxed{+2}) = 2P.$$
  since E_i has ∂_j to right, which is 0 in $\mathbb{C}[x]\otimes C$.
- $(E_1-\beta)\circ P = (1-\beta_1)P, \quad (E_2-\beta)\circ P = (2-\beta_2)P.$
- if $\beta=(1,2)$ both endos are zero, if not one is unit.
- $$\mathcal{H}_\bullet(\beta;C) = \begin{cases} (\mathbb{C}[x],\mathbb{C}[x]\oplus\mathbb{C}[x],\mathbb{C}[x]) & \text{if } \beta=(1,2);\\ (0,0,0) & \text{else.}\end{cases}$$

Rank > vol V: finding jumps

- Feed back into les:
- For $\beta \neq (1,2)$, $\mathrm{rk}(H_A(\beta)) = \mathrm{rk}(\mathcal{H}_0(\tilde{S}_A, \beta)) = \mathrm{vol}(A)$.
- For $\beta = (1,2)$,

$$0 \to \underbrace{\mathcal{H}_1(C;\beta)}_{\text{rk}=2} \to \underbrace{\mathcal{H}_0(S_A;\beta)}_{\text{rk}=?} \to \underbrace{\mathcal{H}_0(\tilde{S}_A;\beta)}_{\text{rk}=4} \to \underbrace{\mathcal{H}_0(C;\beta)}_{\text{rk}=1} \to 0$$

- Rank is additive in exact sequences.
- $\mathrm{rk}(M_A(\binom{1}{2})) = 2 + 4 - 1 = 5 > 4$.

Consequences of Euler–Koszul 9

- $\{\beta \mid \mathrm{rk}(H_A(\beta)) > \mathrm{vol}(A)\} =: \mathcal{E}_A$.
- $\mathcal{E}_A$ =subspace arrangement, each parallel to a face of $\mathbb{R}_+ A$
- $\mathcal{E}_A \ni \beta \Leftrightarrow \beta \in \overline{\deg_A(\mathrm{Ext}^i_{R_A}(S_A, R_A))}^{Zar}$ for some $i > n - d$
- $\mathcal{E}_A \ni \beta \Leftrightarrow \mathcal{K}_\bullet(S_A; \beta)$ not resolution
- $\beta \mapsto \mathrm{rk}(H_A(\beta))$ is upper semi-continuous

Local cohomology and jumps

Setup: $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}$, $\mathfrak{m} = \langle \partial_1, \ldots, \partial_4 \rangle$.

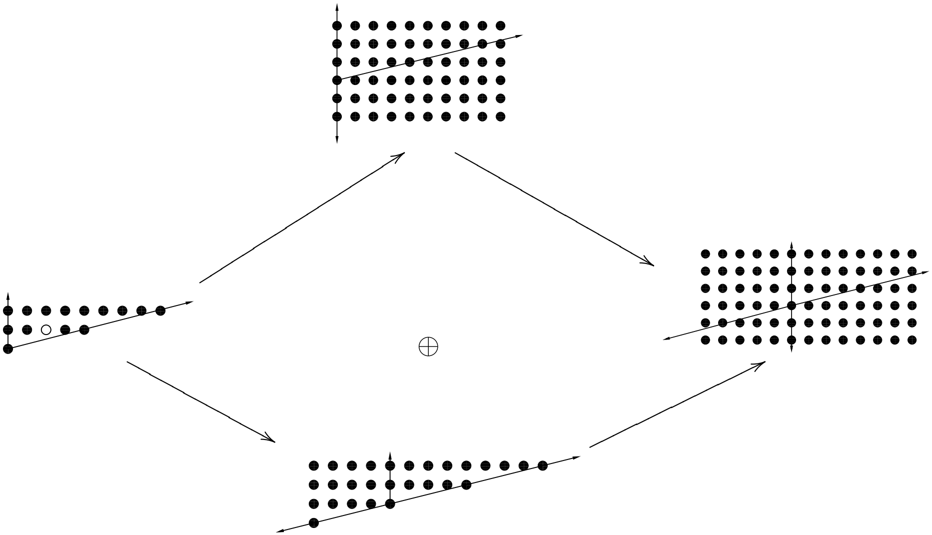
We compute $H^i_{\mathfrak{m}}(S_A)$.

- Fact: $H^i_{\mathfrak{m}}(S_A) = H^i_{\partial_1,\partial_4}(S_A)$ since $\partial_3^3 - \partial_1^2\partial_3, \quad \partial_3^3 - \partial_4^2\partial_1 \in I_A$
- Recall: Čech complex

$$S_A \to S_A[\partial_1^{-1}] \oplus S_A[\partial_4^{-1}] \to S_A[\partial_1^{-1}\partial_4^{-1}].$$

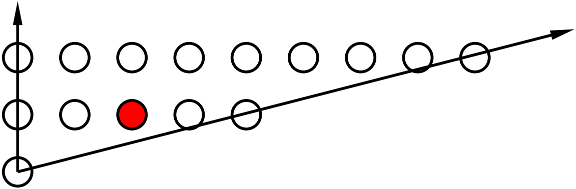
- Each module is graded $\mathbb{C}$-space.
- Local cohomology becomes inclusion/exclusion of dots:

Inclusion/exclusion on pictures 11


$$\oplus$$

Inclusion/exclusion on pictures

Homology: none at left, much at right. Middle:



Note:

$$(H_{\mathfrak{m}}^{<d}(S_A))_\beta \neq 0 \Leftrightarrow \text{rank}>\text{volume in } M_A(\beta).$$

(In general: must take Zariski closure!)

Open questions

- predict size of jump at $\beta \in \mathcal{E}_A$ by algebraic data
- bound $\sup \frac{\mathrm{rk}(H_A(\beta))}{\mathrm{vol}(A)}$ better than by 2^{2d}
- consider $sol(H_A(\beta))$ as function of β.
  - gives constructible sheaves?
  - in terms of local cohomology?
- what is the monodromy around Δ_A?

Some further reading

- Beukers, Heckmann: Inventiones 1989, monodromy of ${}_nF_{n-1}$
- Sturmfels, Saito, Takayama: Springer 2000, book on deformations of D-modules and algorithms
- Stienstra: arXiv:math/0511351, GKZ and applications
- Miller, Matusevich, Walther: JAMS 2005, details on Euler–Koszul
- Passare, Sadykov, Tsikh: Compositio 2005, on singularities
- Schulze, Walther: Duke 2008, irregular solutions to GKZ-systems
- Beukers: Inventiones 2010: algebraic solutions to GKZ
- Dickenstein, Miller, Matusevich: Duke 2010, binomial D-modules
- Castro-Jimenez, Fernandez-Fernandez: TAMS 2011, irregularity
- Takeuchi, Esterov: various articles on monodromy (some to appear)