Homework – Week 1

Math 552, Fall 2018 – Algebraic Geometry

August 27

1. Prove the following (Challenge - do it for power series rings as well).

Theorem. (Hilbert Basis Theorem) If R is Noetherian, then so is $R[x]$.

2. Assume for this problem that $k = \overline{k}$ is an algebraically closed field. Using only properties of UFD's, PID's, and Euclidean domains from a first course in abstract algebra, give an alternate/elementary proof (i.e. without reference to the Nullstellensatz) that the prime ideals of $k[x, y]$ are precisely:
   - 0;
   - $\langle f(x, y)\rangle$ where $f(x, y)$ is an irreducible polynomial;
   - and the maximal ideals $\langle x - a, y - b\rangle$ for $a, b \in k$.

   Explain how to use this description to completely describe the Zariski topology on $k^{\oplus 2}$.
3. Prove the following.

Lemma. The sets of the form $D(f) = k^{\oplus n} \setminus \mathbb{V}(f)$ for $f \in k[x_1, \ldots, x_n]$ form a basis for the Zariski topology on $k^{\oplus n}$. Moreover, the Zariski topology is quasicompact (every open cover has a finite subcover, but it is not necessarily quasicompact).

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4. a. For any field k, identify k^{m^2} with the set of all $m \times m$ matrices. Show that, for all r, the set of matrices with rank $\leq r$ is an affine algebraic subset of k^{m^2}.

   b. Show that the set $SU(2)$ of all complex 2×2 unitary matrices with determinant one is not an affine algebraic subset of $\mathbb{C}^4$, but that it can be thought of as an affine algebraic subset of $\mathbb{R}^8$.
5. Find $\mathbb{I}(X)$ where $X = \mathbb{V}(x^2, xy^2) \subseteq k^2$. Check that it is the radical of $\langle x^2, xy^2\rangle$. What are the irreducible components of X?
6. Suppose that $X \subseteq k^n$ is an algebraic set. Show that X is irreducible if and only if $\mathbb{I}(X)$ is a prime ideal.

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7. Prove the determinant trick: every endomorphism of a finitely generated module over a commutative ring satisfies a monic relation.
8. Prove Zariski's lemma: if $R \subseteq S$ is a module-finite extension of domains, then R is a field if and only if S is a field.