Homework – Week 2

Math 552, Fall 2018 – Algebraic Geometry

September 5

1. Assume $k = \overline{k}$ is an algebraically closed field. Show that there is a bijective correspondence between projective algebraic sets in $\mathbb{P}^n_k$, and radical homogeneous ideals in $k[x_0, \ldots, x_n]$ excluding the irrelevant ideal $(x_0, \ldots, x_n)$.
2. Let k be any field, $\mathbb{P}^n_k = U_0 \cup \cdots \cup U_n$ the standard affine cover of projective space, and $X \subseteq \mathbb{P}^n$ any subset. Show that X is a projective algebraic set (*i.e.* $X = \mathbb{V}(\Sigma)$ for some set Σ of homogenous polynomials in $k[x_0, \ldots, x_n]$) if and only if $X \cap U_i$ is an affine algebraic subset of $U_i \cong k^n$ for each $i = 0, \ldots, n$.

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3. Consider an affine algebraic set $X \subseteq k^n \cong U_0 \subseteq \mathbb{P}^n_k$, and let $\overline{X}$ be the corresponding projective closure. Show that
$$\overline{X} = \mathbb{V}(\{\tilde{f} \mid f \in \mathbb{I}(X)\})$$
where $\tilde{f}$ denotes the homogenization of a polynomial $f \in k[x_1, \ldots, x_n]$ with respect to x_0.
4. Complete the following.
   (a) Describe the image of the regular map $k^2 \to k^2$ given by $(x, y) \mapsto (x, xy)$. Is it open? closed? dense?
   (b) Assume the characteristic of k is not 2. For $c \in k$, let V_c denote the intersection of the affine algebraic sets $V = \mathbb{V}(x^2 + y^2 - 1)$ and $\mathbb{V}(x - c)$ in k^2. Find $\mathbb{I}(V_c)$. Two choices of c are special, which ones? Consider the polynomial map given by $V \to k$ given by $(x, y) \mapsto x$. "Draw" the map and interpret the special c's geometrically (these are called ramification points).
   (c) What happens to (b) in characteristic 2?
   (d) Give the induced maps on coordinate rings in (a) and (b).