Homework – Week 3

Math 552, Fall 2018 – Algebraic Geometry

September 10

1. Suppose that $k = \overline{k}$ is algebraically closed.
   (a.) Show that the Zariski topology on k^2 is not the product topology on $k \times k$.
   (b.) If $X \subseteq k^n$ and $Y \subseteq k^m$ are affine algebraic sets, and we view $X \times Y$ as an affine algebraic set in k^{n+m}, show that $k[X \times Y] \cong k[X] \otimes_k k[Y]$ and deduce that $X \times Y$ is a categorical product.
   (c.) If $X, Y \subseteq k^n$ are disjoint affine algebraic sets, show that $k[X \cup Y] \cong k[X] \times k[Y]$.
2. Suppose the R and S are arbitrary commutative rings.
   (a.) Show that every irreducible closed subset X of $\mathrm{Spec}(R)$ is the closure of a uniquely determined point, which is in fact $\mathbb{I}(X)$ and called the generic point.
   (b.) Let $R \to S$ be a ring homomorphism. Show that it induces a continuous map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ defined by $Q \mapsto Q \cap R$.
3. (a.) Let $\pi\colon Y \to X$ be a continuous map of topological spaces. If $Z \subseteq Y$ is irredicuble, then so is $\pi(Z) \subseteq X$.
   (b.) Let X be a topological space. If $Z \subseteq X$ is irreducible, then so is its closure $\overline{Z}$.
   (c.) Let $\pi\colon Y \to X$ be a regular map of affine algebraic sets. Show that $\pi(Y)$ is dense in X (*i.e.* π is dominant) if and only if the induced map $k[X] \to k[Y]$ on coordinate rings is injective.

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4. Prove that a UFD is normal.
5. State and prove the first three Cohen-Seidenberg Theorems governing the behavior primes in integral extensions: lying over, incomparability, and going up. (Challenge: also prove going down.)