Math 552 Algebraic Geometry I
Lecture #2 Recap

August 29, 2018

Exercise.

(a) Let k be any field. Identify $k^{\oplus m^2}$ with the set of all $m \times m$ matrices. Show that the set of matrices of rank $\leq r$ is an algebraic set.

(b) Show that SU(2), the set of 2×2 complex unitary matrices, is not an algebraic subset of $\mathbb{C}^4$. On the other hand, show that it is an algebraic subset of $\mathbb{R}^8$.

Definition. Let k be a field, and let $S \subseteq k^n$ be **any** subset. $\mathbb{I}(S) = \{f \in k[x_1, \ldots, x_n] \mid f(\underline{\alpha}) = 0 \text{ for all } \underline{\alpha} \in S\}$.

Lemma. *Let k be a field.*

(1) If $X \subseteq k^n$ is an algebraic set, then $X = \mathbb{V}(\mathbb{I}(X))$, $\mathbb{I}(X)$ is the largest ideal with this property, and it is radical.

(2) If $S \subseteq k^n$ is any subset, then $\mathbb{V}(\mathbb{I}(S)) = \overline{S}$, the closure of S in the Zariski topology.

Definition. Let X be a topological space. We say that X is *reducible* if it is the union of two proper closed subsets; that is, if $X = X_1 \cup X_2$ with $X_i \subsetneq X$ closed. Otherwise, we say that X is *irreducible*, i.e. if $X = X_1 \cup X_2$, then $X = X_1$ or $X = X_2$.

Remark. If $Y \subseteq X$ is a closed subspace of a topological space X, we speak of Y being irreducible with respect to the subspace topology; that is, Y is irreducible if and only if $Y \subseteq X_1 \cup X_2$ with $X_i \subseteq X$ closed implies that $Y \subseteq X_1$ or $Y \subseteq X_2$.

Lemma. *Let k be a field and let $X \subseteq k^n$ be an algebraic set. Then X is irreducible if and only if $\mathbb{I}(X)$ is a prime ideal.*

Exercise. Prove the preceding lemma.

Theorem. *Let k be a field and let $X \subseteq k^n$ be an algebraic set. Then $X = X_1 \cup \cdots \cup X_m$ where the X_i are the maximal irreducible closed subsets of X. Moreover, this decomposition is unique up to ordering of the factors.*

Exercise. Let k be an algebraically closed field. Find $\mathbb{I}(X)$, where $X = \mathbb{V}(x^2, xy) \subseteq k^{\oplus 2}$. Check that $\mathbb{I}(X) = \sqrt{(x^2, xy)}$ explicitly, and identify the irreducible components of X.

Theorem (Hilbert's Nullstellensatz). *Let k be an algebraically closed field.*

(Weak) *Every maximal ideal of $k[x_1, \ldots, x_n]$ has the form $(x_1 - \alpha_1, \ldots, x_n - \alpha_n)$ for $\underline{\alpha} \in k^n$.*

(Strong) *Let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal. Then $\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}$.*

Corollary. *Let k be an algebraically closed field. Then there is an inclusion-reversing bijection*

$$\{\text{(irreducible) algebraic subsets of } k^n\} \longleftrightarrow \{\text{(prime) radical ideals of } k[x_1, \ldots, x_n]\}$$

given by

$$X \longmapsto \mathbb{I}(X)$$

and

$$\mathbb{V}(J) \longleftarrow J.$$

Corollary. *Let k be an algebraically closed field. Then every radical ideal $I \subseteq k[x_1, \ldots, x_n]$ can be expressed as $I = P_1 \cap \cdots \cap P_m$ where the P_i are (finitely many) minimal primes over I.*

Remark. Note that the above statement is true for any Noetherian ring, but this does not follow directly from the Nullstellensatz.