MATH 552 ALGEBRAIC GEOMETRY I

AUGUST 31 RECAP

Definition (Coordinate Ring) Let $X \subseteq k^n$ be some algebraic set. Define

$$k[X] \coloneqq \{\varphi : X \to k | \varphi = f|_X \text{ for some } f \in k[x_1, \ldots, x_n]\}.$$

Lemma

$$k[x_1, \ldots, x_n] \twoheadrightarrow k[X] \text{ via } f \mapsto f|_X$$

with kernel $\mathbb{I}(X)$. Thus

$$k[X] \cong k[x_1, \ldots, x_n]/\mathbb{I}(X).$$

Note $k[X]$ is a finitely generated k-algebra with generators $x_i|_X$, the restrictions of the coordinate functions to X, hence the name Coordinate Ring.

Definition(Finite, Finite Type, and Integral) Let φ be a ring map $\varphi : R \to S$. We say that the map is of finite type if S is a finitely generated R-algebra, i.e. we have the ring isomorphism

$$S \cong R[x_1, \ldots, x_n]/(\text{some ideal}).$$

We say that the map is finite if S is a finitely generated R-module, i.e. we have the R-module isomorphism

$$S \cong R^{\oplus n}/(\text{some } R\text{-submodule}).$$

We say that the map is integral if every element of S satisfies a monic polynomial over the image of R, i.e. given $s \in S$, there exist elements $a_i \in R$ such that

$$s^n + a_1 s^{n-1} + \cdots + a_n = 0,$$

where we understand $a_i s^m$ to mean $\varphi(a_i)s^m$ in the case that φ is not an inclusion.

Lemma Finite = Finite Type + Integral

Theorem(Noether Normalization Lemma) Let k be any field and R be a finite type k-algebra. Then R is module finite over a polynomial subring, i.e. there exist elements $z_i \in R$ such that

$$k[z_1, \ldots, z_n] \subseteq R$$

and the inculsion map is finite.

Lemma(Zariski's Lemma) Let $R \subseteq S$ be an integral extension of domains. Then R is a field if and only if S is a field.

Theorem(Weak Nullstellensatz) Let k be any field. If $M \subseteq k[x_1, \ldots, x_n]$ is a maximal ideal, then $k \subseteq k[x_1, \ldots, x_n]/M$ is a finite algebraic extension of k. In particular, if k is algebraically closed, we get $k[x_1, \ldots, x_n]/M \cong k$ and $M = (x_1 - \alpha_1, \ldots, x_n - \alpha_n)$ for some $\alpha_i \in k$.