Math 552, Algebraic Geometry
November 14, 2018 Lecture Recap

Note from the scribe. All categories will be underlined, e.g set. Schemes are not assumed to be separated, i.e., I will not use the word "prescheme." Sometimes I will denote the category of schemes by sch.

Motivation. We will shortly present the functor of points of a scheme, but first here is some motivation. Let $X \subseteq \mathbb{A}_k^n$ be a closed subset defined by some equations $f_1, ..., f_r$, where $k = \bar{k}$. A point in X can be given by an n-tuple $\alpha = (\alpha_1, ..., \alpha_n) \in k^n$ such that $f_i(\alpha) = 0$ for all i. Analogously, for inclusions of fields $k_0 \subseteq \ell \subseteq k$, an ℓ-point of X is given by an n-tuple $\beta = (\beta_1, ..., \beta_n) \in \ell^n$ such that $f_i(\beta) = 0$ for all i. In light of this, we establish the following working definition: for S a k_0-algebra, an S-point of X is given by $\alpha \in S^{\oplus n}$ such that $f_i(\alpha) = 0$ for all i. Observe further that giving an S-point of X is equivalent to giving a k_0-algebra homomorphism $k_0[X] \to S$ taking (the image of) $x_i \mapsto \alpha_i$. This motivates the following definition.

Definition 1. *Let $Z, X \in \underline{scheme}$ be given. A Z-point of X is a morphism $Z \to X$. We will sometimes write $Z(X) = Hom_{\underline{sch}}(Z, X)$ for the set of Z-points of X.*

Example 2. *In set, given maps $\pi_1, \pi_2 : X_1, X_2 \to Y$, we obtain the pullback $X_1 \times_Y X_2 = \{(x_1, x_2) \in X_1 \times X_2 \mid \pi_1(x_1) = \pi_2(x_2)\}$. That is, a choice of X_1-point and X_2-point of Y determines a $X_1 \times_Y X_2$-point of Y.*

Exercise 3. *In scheme, $Z(X_1 \times_Y X_2) = \{(f, g) \in Z(X_1) \times Z(X_2) \mid f, g$ determine the same Z-point of $Y\}$.*

Definition 4. *Let $X \in \underline{scheme}$ be given. The functor of points h_X of X is the functor assigning $h_X : Z \mapsto Hom_{\underline{sch}}(Z, X)$. It is a contravariant functor to set.*

We establish some notation before stating the next proposition. For categories $\mathscr{C}, \mathscr{D}$, we denote by $F(\mathscr{C}, \mathscr{D})$ the functor category of $\mathscr{C}$ and $\mathscr{D}$, whose objects are functors $\mathscr{C} \to \mathscr{D}$ and whose morphisms are natural transformations between these functors.

Proposition 5 (Yoneda's lemma)**.** *Let $X_1, X_2 \in \underline{scheme}$ be given. Then there is bijection*

$$Hom_{\underline{sch}}(X_1, X_2) \simeq Hom_{F(\underline{sch}, \underline{set})}(h_{X_2}, h_{X_1}).$$

The proof of this proposition (is short and) can be found in any standard reference on category theory or algebraic geometry.

Exercise 6. *In particular, show that $X_1 \simeq X_2$ (as schemes) if and only if $h_{X_1} \simeq h_{X_2}$ (as functors).*

Note that we may restrict our attention to *affine* points of X:

Exercise 7. *Define*

$$h'_X : \underline{ring} \longrightarrow \underline{set}$$
$$R \longmapsto Hom_{\underline{sch}}(Spec(R), X).$$

Show that $h'_{X_1} \simeq h'_{X_2}$ if and only if $X_1 \simeq X_2$, cf. Exercise 6 above.

Lemma 8. *Let k be a field. A k-point of a scheme X is given by (1) a (set-theoretic) point $x \in X$, and (2) an inclusion $\kappa(x) \to k$ (cf. Hartshorne's II.2.7).*

Proof sketch. We write $X = \cup U_i$ for a covering by open affine subschemes, with $U_i \simeq \operatorname{Spec}(R_i)$. Then a map $\operatorname{Spec}(k) \to X$ is given by a family of compatible ring maps $R_i \to k$ for all U_i containing x, where x corresponds in R_i to the kernel $\mathfrak{p}_i$ of each of these maps. By the mapping properties of the localization and kernel, this induces a map $\kappa(x) = (R_i)_{\mathfrak{p}_i}/\mathfrak{p}_i(R_i)_{\mathfrak{p}_i} \to k$ which is injective, and this is reversible. □

Example 9. *Spec$\left(\frac{\mathbb{R}[x]}{(x^2+1)}\right)$ has two $\mathbb{C}$-points, and only one set-theoretic point.*

Lemma 10. *Let $(R, \mathfrak{m})$ be a local ring. An R-point of a scheme X is determined by (1) a (set-theoretic) point $x \in X$, and a local homomorphism of local rings $\mathcal{O}_{X,x} \to R$.*

Exercise 11. *Let X be a scheme of finite type over a field $k = \bar{k}$. Set $\epsilon =$ Spec$\left(\frac{k[t]}{(t^2)}\right)$, the ring of dual numbers of k. Show that an ϵ-point of X over k is given by (1) a choice of closed point $x \in X$, and (2) a choice of element of $(\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$, where $\mathfrak{m}_x$ is the maximal ideal corresponding to x in some Spec$(R) \subseteq X$ (note that the ideal is indeed maximal, as the point x is assumed to be closed).*

Note. We will eventually define $(\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$ to be the *Zariski tangent space* of X at x, so the above exercise says that determining an ϵ-point of X means picking a point x of X and a tangent direction at x.