# Math 552, Algebraic Geometry
## November 2, 2018 Lecture Recap

Today's was a guest lecture from Izzet. We computed the Hilbert polynomial in several common projective settings.

**Example 1** (Points). *Let $X$ be a set of $k$ distinct points in $\mathbb{P}^n$. Then $p_X(m) = k$.*

**Example 2** (Hypersurfaces). *Let $X \subseteq \mathbb{P}^n$ be a hypersurface $X = V(F)$ defined by the homogeneous polynomial $F$ of degree $d$. We can see that $\deg(X) = d$ by intersecting $X$ with a line, and appealing to the algebraic closure of the base field $k$.*

*Alternatively, we examine the Hilbert polynomial. We have, for $m \gg 0$,*

$$\begin{aligned}
\dim_k\left(\frac{k[x_0,\dots,x_n]}{(F)}_m\right) &= \dim(\textit{polynomials of degree } m) - \dim(\textit{polynomials of degree } m \textit{ divisible by } F)\\
&= \binom{n+m}{n} - \binom{m-d+n}{n}\\
&= \frac{(m+n)(m-1+n)\cdots(1+m)}{n!} - \frac{(m-d+n)\cdots(m-d+1)}{n!}\\
&= \frac{m^n}{n!} + \frac{m(n+1)}{2}\frac{m^{n-1}}{n!} + \textit{lower order terms}\\
&\quad - \left(\frac{m^n}{n!} + \left(\frac{m(n+1)}{2} - dn\right)\frac{m^{n-1}}{n!} + \textit{lower order terms}\right)
\end{aligned}$$

*so it follows that*

$$h_X(m) = \left[\frac{dn}{n!}m^{n-1} = \frac{d}{(n-1)!}m^{n-1}\right] + \textit{lower order terms}$$

*so that, again,* $\deg(X) = d$.

**Example 3** (Linear subspaces). *Let $X \simeq \mathbb{P}^k \subseteq \mathbb{P}^n$ be a linear subspace. Then* $\deg(X) = 1$. *Indeed, choose homogeneous coordinates on $\mathbb{P}^n$ such that $X = V(x_{k+1},\dots,x_n)$. Then for $m \gg 0$,*

$$\dim(k[x_0,\dots,x_k]_m) = \binom{k+m}{k} = \frac{(k+m)(k-1+m)\cdots(1+m)}{k!} = \frac{1}{k!}m^k + \textit{lower order terms}$$

*so again* $\deg(X) = 1$.

**Example 4** (Irreducible curves). *Let $C \subseteq \mathbb{P}^n$ be an irreducible curve. The arithmetic genus of $C$ (defined in the preceding lecture) is $1 - p_C(0)$. If $\deg(C) = d$, with arithmetic genus $g$, for $m \gg 0$*

$$p_C(m) = dm + 1 - g.$$

*Quote from I.C.: "Always remember this."*

*So when $C$ is a plane curve of degree $d$, as in the second example,*

$$\begin{aligned} h_C(m) &= \binom{m+2}{2} - \binom{m-d+2}{2} \\ &= \frac{(m+2)(m+1)}{2} - \frac{(m-d+2)(m-d+1)}{2} \\ &= dm + 1 - \binom{d-1}{2} \end{aligned}$$

*so that the arithmetic genus of $C$ is $\binom{d-1}{2}$.*

**Example 5** (Veronese embeddings)**.** *For $X \subseteq \mathbb{P}^n \overset{\nu_d}{\hookrightarrow} \mathbb{P}^{\binom{n+d}{d}-1}$ the Veronese embedding of $X$, we have*

$$p_{\nu_d(X)}(m) = p_X(dm)$$

*and substantiating this is a homework exercise.*

*As a sub-example, the rational normal curve $X$ of degree $d$ has degree $d$, as*

$$h_X(m) = dm + 1$$

*so* $\deg(X) = d$.

**Example 6** (Segre embeddings)**.** *Let $\Sigma : \mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^{(n+1)(m+1)-1}$ be the Segre embedding and $\Sigma_{n,m}$ be its image, the Segre variety. Then*

$$h_{\Sigma_{n,m}}(d) = \binom{d+n}{n} - \binom{d+m}{m}$$

*so* $\deg(\Sigma_{n,m}) = \frac{(m+n)!}{m!n!}$.

**Example 7** (Grassmannians)**.** *Let $P : G(2,n) \hookrightarrow \mathbb{P}(\wedge^2 k^n)$ be the Plücker embedding. Quote from I.C.: "Tell Kevin all fields are just the complex numbers." Then it turns out the degree of $G(2,n)$ is the nth Catalan number. (the case $n = 4$ can be checked by work from a previous homework assignment: $G(2,4)$ is cut out by a single quadric.)*