MATH 552 Algebraic Geometry I

Nov 21 Recap

Throughout this recap, X is a scheme of finite type over a field k with $k = \overline{k}$, and $x \in X$ is a closed point.
Our goal is to define the Zariski tangent space to X at x, denoted by T_xX.
T_xX is a local notion, so we will assume that $X = \mathbb{V}(I) \subseteq \mathbb{A}^n_k$.
Moreover, we may choose coordinates so that $x = (0, ..., 0) \in \mathbb{A}^n_k$.

Definition. ($df|_0$)
Given $f \in k[x_1, ..., x_n]$, define $df|_0 := \sum_{i=1}^n \frac{\partial f}{\partial x_i}(0)x_i$
Remarks.
(a). $df|_0$ =(linear part of f in the Taylor expansion of f at 0).
(b). $df|_0 = f_1$, where $f = f_0 + f_1 + \cdots + f_d$ is a decomposition of f as the sum of its homogeneous components.
(c). $d(fg)|_0 = g(0)df|_0 + f(0)dg|_0$ for all $f, g \in k[x_1, ..., x_n]$.

Four definitions of T_0X**.**
(1). T_0X :=(union of lines through 0 in k^n that are tangent to X at 0). (Notion of tangent lines is defined later.)
(2). $T_0X := \mathbb{V}\big(df|_0 \,\big|\, f \in I\big) \subseteq k^n$.
(3). $T_0X := \big(\mathfrak{m}_{X,0}/\mathfrak{m}^2_{X,0}\big)^\vee := \mathrm{Hom}_k\left(\mathfrak{m}_{X,0}/\mathfrak{m}^2_{X,0}, k\right)$.
(4). $T_0X := \mathrm{Der}_k(\mathcal{O}_{X,0}, k) := \big\{$point derivations $d : \mathcal{O}_{X,0} \to k$ (i.e. k-linear maps with $d(fg) = f(0)dg + g(0)df$)$\big\}$.
Remarks.
(a). We will show that these 4 definitions give isomorphic vector spaces.
(b). (3) allows us to define T_xX intrinsically: For any closed point $x \in X$, define $T_xX := \big(\mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x}\big)^\vee$.
(c). In general, we have

$$\begin{aligned} \dim_k T_xX &= \dim_k \mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x} \\ &= \text{minimum \# of generators of } \mathfrak{m}_{X,x} \quad \text{(Nakayama)} \\ &\geqslant \dim \mathcal{O}_{X,x} \quad \text{(Krull's Hauptidealsatz)} \end{aligned}$$

Definition. (non-singular/smooth at x)
X is said to be **non-singular/smooth** at x if $\dim_k T_xX = \dim \mathcal{O}_{X,x}$.
($\Leftrightarrow \dim_k \mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x} = \dim \mathcal{O}_{X,x} \overset{\triangle}{\Leftrightarrow} \mathcal{O}_{X,x}$ is a **regular local ring**.)
Remark.
When X is reduced and irreducible, $\dim_k \mathcal{O}_{X,x} = \dim X$ by x is closed.
Then, we have $\dim_k T_xX \geqslant \dim X$ in general, and that the equality holds iff X is non-singular at x.

Definition. (tangent lines)

Given $\alpha := (a_1, ..., a_n) \in k^n \backslash \{0\}$, let $l_\alpha := \{t\alpha \mid t \in k\}$ be the line through 0 and α.

For all $f \in I$, write $f = f_0 + f_1 + \cdots + f_d$ (sum of homogeneous components). Note that $f_0 = 0$ by $0 \in X$.

We have $f(t\alpha) = tf_1(\alpha) + \cdots + t^d f_d(\alpha)$.

Now, let n be the maximum power of t such that $t^n | f(t\alpha)$ for all $f \in I$.

Then, l_α is said to be **tangent to** X **at** 0 if $n \geqslant 2$.

Examples of T_0X **(in the sense of (1)).**

Consider $X := \mathbb{V}(y - x^2) \subseteq \mathbb{A}^2$.

l_α with $\alpha = (a_1, a_2)$ is tangent to X at 0 $\Leftrightarrow t^2 \big| ta_2 - t^2 a_1^2 \Leftrightarrow a_2 = 0$

Thus, $T_0X = x$-axis.

Next, consider $Y := \mathbb{V}(y^3 - x^3 - x^2) \subseteq \mathbb{A}^2$.

l_α with $\alpha = (a_1, a_2)$ is tangent to Y at 0 $\Leftrightarrow t^2 \big| t^2 a_2^2 - t^3 a_1^3 - t^2 a_1^2$. This is true for all α.

Thus, $T_0Y = \mathbb{A}^2$.

Proof of $(1) \Leftrightarrow (2)$**.**

Since $df|_0 = f_1$, we have l_α is tangent to X at 0 $\Leftrightarrow f_1(\alpha) = df|_0(\alpha) = 0$ for all $f \in I \Leftrightarrow l_\alpha \subseteq \mathbb{V}\big(df|_0 \,\big|\, f \in I\big)$.

Lemma

Let $X = \mathbb{V}(I) \subseteq \mathbb{A}_k^n$ and suppose $0 \in X$. If $I = (f_1, ..., f_r)$, then $\mathbb{V}\big(df|_0 \,\big|\, f \in I\big) = \mathbb{V}\big(df_i|_0 \,\big|\, i = 1, ..., r\big)$.

Hint: Given $f \in I$ and $g \in k[x_1, ..., x_n]$, $d(gf)|_0 = g(0) df|_0$

Theorem. (Jacobian criterion for non-singularity).

Let $X = \mathbb{V}(f_1, ..., f_r) \subseteq \mathbb{A}_k^n$ be reduced and irreducible, and suppose $0 \in X$.

Then X is non-singular at 0

$\Leftrightarrow$ rank $\left[\frac{\partial f_i}{\partial x_j}(0)\right]_{i,j} = n - \dim X$

$\Leftrightarrow$ some $(n - \dim X) \times (n - \dim X)$ minor of $\left[\frac{\partial f_i}{\partial x_j}(0)\right]_{i,j}$ has nonzero determinant.

The matrix $\left[\frac{\partial f_i}{\partial x_j}(0)\right]_{i,j}$ is given by

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1}(0) & \cdots & \frac{\partial f_1}{\partial x_n}(0) \\ \vdots & & \vdots \\ \frac{\partial f_r}{\partial x_1}(0) & \cdots & \frac{\partial f_r}{\partial x_n}(0) \end{bmatrix}.$$

Proof.

In the sense of (2), $T_0X = \ker \left[\frac{\partial f_i}{\partial x_j}(0)\right]_{i,j}$. Thus, $\dim_k T_0X = n - \text{rank} \left[\frac{\partial f_i}{\partial x_j}(0)\right]_{i,j}$.

Since X is reduced, irreducible, and non-singular at 0, $\dim_k T_0X = \dim X$, implying rank $\left[\frac{\partial f_i}{\partial x_j}(0)\right]_{i,j} = n - \dim X$.

Remark.

In general, we have rank $\left[\frac{\partial f_i}{\partial x_j}(0)\right]_{i,j} \leqslant n - \dim X$. The equality holds iff X is non-singular at 0.

Example.

Let $0 \in X = \mathbb{V}(f) \subseteq \mathbb{A}^n$ be a hypersurface, where $f \in k[x_1, ..., x_n]$ is irreducible.

Then, X is non-singular at 0 $\Leftrightarrow \left[\frac{\partial f}{\partial x_1}(0) \cdots \frac{\partial f}{\partial x_n}(0)\right]$ has a nonzero entry.

Proof of $(2) \Leftrightarrow (3)$**.**

Let $\eta := (x_1, ..., x_n) \subseteq k[x_1, ..., x_n]$. Note that $I \subseteq \eta$ by $0 \in X$ and that $\mathcal{O}_{X,0} = k[X]_{\eta k[X]}$.

$\mathfrak{m}_{X,0}/\mathfrak{m}_{X,0}^2 \cong (\eta/I)/\big((\eta^2+I)/I\big) \cong \eta/(\eta^2+I) \cong (\eta/\eta^2)/\big((\eta^2+I)/\eta^2\big)$.

$\eta/\eta^2 \cong$ (linear forms of k^n) $= (k^n)^\vee$.

$(\eta^2+I)/\eta^2 \cong \big($subspace of $(k^n)^\vee$ generated by $df|_0$ $(f \in I)\big)$.

$\Rightarrow \mathfrak{m}_{X,0}/\mathfrak{m}_{X,0}^2 \cong (k^n)^\vee/\big($subspace generated by $df|_0$ $(f \in I)\big) \cong (T_0X)^\vee$ in the sense of (2) $\Rightarrow T_0X \cong \big(\mathfrak{m}_{X,0}/\mathfrak{m}_{X,0}^2\big)^\vee$

Lemma.

If $d \in \mathrm{Der}_k(\mathcal{O}_{X,0}, k)$, then $\mathfrak{m}_{X,0}^2 \subseteq \ker d$.

Proof.

Let $f, g \in \mathfrak{m}_{X,0}$. By $f(0) = g(0) = 0$, $d(fg) = f(0)dg + g(0)df = 0$.

Proof of $(3) \Leftrightarrow (4)$**.**

Given $d \in \mathrm{Der}_k(\mathcal{O}_{X,0}, k)$, we may define $\tilde{d} : \mathfrak{m}_{X,0}/\mathfrak{m}_{X,0}^2 \hookrightarrow \mathcal{O}_{X,0}/\mathfrak{m}_{X,0}^2 \xrightarrow{\bar{d}} k$.

$d \mapsto \tilde{d}$ gives us a map $\mathrm{Der}_k(\mathcal{O}_{X,0}, k) \to \big(\mathfrak{m}_{X,0}/\mathfrak{m}_{X,0}^2\big)^\vee$.

On the other hand, given $\phi \in \big(\mathfrak{m}_{X,0}/\mathfrak{m}_{X,0}^2\big)^\vee$, we can get $d\phi : \mathcal{O}_{X,0} \to k$. (Check: this map is linear.)

by composing $\mathcal{O}_{X,0} \to \mathfrak{m}_{X,0} \twoheadrightarrow \mathfrak{m}_{X,0}/\mathfrak{m}_{X,0}^2 \xrightarrow{\phi} k$, where $\mathcal{O}_{X,0} \to \mathfrak{m}_{X,0}$ is given by $f \mapsto f - f(0)$.

(Check: $d\phi \in \mathrm{Der}_k(\mathcal{O}_{X,0}, k)$ and that the map $\phi \mapsto d\phi$ is an inverse of $d \mapsto \tilde{d}$.)

Remark.

$(3) \Leftrightarrow (4)$ canonically.