MATH 552 ALGEBRAIC GEOMETRY I

NOVEMBER 26 RECAP

Jacobian Criterion Let $k = \bar{k}$, and suppose $0 \in X = \mathbb{V}(I) = \mathbb{V}(f_1, f_2, \dots) \subseteq \mathbb{A}^n_k$. Then $0 \in X$ is a smooth/nonsingular point iff the rank of the Jacobian matrix equals $\operatorname{codim}(X \subseteq \mathbb{A}^n)$.

Example If $X = \mathbb{V}(f)$ a hypersurface, then $\mathbb{V}(\frac{\partial f}{\partial x_1}(0)x_1 + \cdots + \frac{\partial f}{\partial x_n}(0)x_n) = T_0X$.

N.B. The locus of smooth points is open. Indeed, in general, if $X = \mathbb{V}(I)$, then $X_{sing} = \mathbb{V}(I, \text{all codim sized minors of Jacobian matrix})$.

Example If $X = \mathbb{V}(f)$, then $X_{sing} = \mathbb{V}(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$.

N.B. It can happen that $X = X_{sing}$. Indeed, if $X = \mathbb{V}(x^2)$ as a scheme in $\mathbb{A}^1_k$, then $X_{sing} = \mathbb{V}(x^2, 2x) = X$. Note that X was not reduced.

Theorem If X is an irreducible variety over an algebraically closed (or just perfect) field, then X has smooth points which are closed. In fact, we see from the Jacobian criterion that it has a nonempty open subset of them.

This can be shown via the following:

Lemma If X is a scheme and $\mathcal{F}$ is a coherent sheaf on X, then the locus of points $x \in X$ where $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module is open.

Example If f is separable over k and we set $K = k[t]/(f)$, then $\Omega_{K/k} = k[t]/(f, f') = (0)$.