Algebraic Geometry I 11-28 Recap

Def. (Projective Tangent Space)

Let $X \subseteq \mathbb{P}^n_k$, $\alpha \in X$. Say $\alpha \in U_i = \mathbb{P}^n_k \setminus \mathbb{V}(x_i)$,

$U_i \cong \mathbb{A}^n_k$.

Then $T_\alpha X \subseteq U_i$

The projective tangent space of X at α is the Zariski closure

$$\overline{T_\alpha X} \subseteq \mathbb{P}^n_k .$$

Prop. (Projective Jacobian Criterion)

$\overline{T_\alpha X}$ = linear subspace of $\mathbb{P}^n$ determined by vanishing of linear forms of projective Jacobian.

Say $X = \mathbb{V}(I)$, I homogeneous

For $(f_1, \cdots, f_s)$ homogeneous set of generators of I, the projective Jacobian at $\alpha \in X$ is:

$$\begin{bmatrix} \partial f_1/\partial x_0 (\alpha) & \cdots & \partial f_1/\partial x_n (\alpha) \\ \vdots & \ddots & \vdots \\ \partial f_s/\partial x_0 (\alpha) & \cdots & \partial f_s/\partial x_n (\alpha) \end{bmatrix},$$

and $\overline{T_\alpha X} = \mathbb{V}\left(\sum_i \partial f/\partial x_i (\alpha) \cdot x_i \text{ for all } f \in I\right)$.

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$\overline{X_{sing}} = \mathbb{V}(I,$ codim-sized minors of Jacobian $)$

(not evaluated at α)

example.

$$X = \mathbb{V}(x^3 - y^2 z) \subseteq \mathbb{P}^2.$$

Jacobian: $\begin{bmatrix} 3x^2 & -2yz & -y^2 \end{bmatrix}$

- at $[0:0:1]\in X$, the Jacobian is $[0:0:0]$, hence has kernel $\mathbb{P}^2$,

  so $\overline{T_{[0:0:1]}X} = \mathbb{P}^2$.

- at $[1:1:1]\in X$, the Jacobian is $\begin{bmatrix} 3 & -2 & -1 \end{bmatrix}$

  so $\overline{T_{[1:1:1]}X} = \mathbb{V}(3x - 2y - z)$.

In the affine patch $z=1$, $X = \mathbb{V}(x^3 - y^2)$, and the Jacobian at $(1,1)$ is $\begin{bmatrix} 3 & -2 \end{bmatrix}$

so $T_{(1,1)}(X|_{z=1}) = \{(x,y) \mid 3(x-1) - 2(y-1) = 0\}$.

Gauss Map.

Let $X \subseteq \mathbb{P}^n$, smooth quasiprojective of dimension d.

Let $x \in X$. $\mathbb{T}_x X$ (projective) $\subseteq \mathbb{P}^n$ is a linear subspace of dimension d.

The Gauss map is the map

$$X \xrightarrow{\pi} G(d,n)$$
$$x \longmapsto \mathbb{T}_x X \quad .$$

Claim. π is regular.

Exercise show $G(d,n) \cong G(n-d,n)$

Projective Tangent Bundle

Let $X \subseteq \mathbb{P}^n$ smooth quasiprojective of dimension d.

Let $\Gamma = \{(x,[w]) \mid w = \mathbb{T}_x X\}$ the graph of π.

The tangent bundle of X is

$$\mathbb{T}X = \{(\alpha, \beta) \mid \beta \in \mathbb{T}_\alpha X\}$$

$$\mathbb{T}X \underset{\text{closed}}{\subseteq} X \times \mathbb{P}^n \cong \Gamma \times \mathbb{P}^n.$$

TX is irreducible of dimension $2d$

each fiber of $TX \to X$ is isomorphic to $\mathbb{P}^d$.