MATH 552 Algebraic Geometry I

Nov 5 Recap

Let R be a commutative ring with 1. As a set, $X = Spec\ R = \{prime\ ideals\ of\ R\}$. Moreover, $Spec\ R$ is a topological space.

· **Closed sets**: $\mathbb{V}(\Sigma) = \{p \in Spec\ R | \Sigma \subset P\}$ for some $\Sigma \subset R$.

· **Basis of the topology**: $\{X_g | g \in R\}$, where $X_g = X \backslash \mathbb{V}(g)$.

· **Quasicompact**: $X = \cup_\alpha X_{g_\alpha} \Leftrightarrow 1 \in (\ldots, g_\alpha, \ldots) \Leftrightarrow 1 = \sum_{i=1}^{l} f_i g_i$ where l is finite. N.B. Open subset of X may not be quasicompact when R is not Noetherian.

· **Points are not always closed**. $p \in X$ is closed $\Leftrightarrow p \subset R$ is maximal ideal.

· **Every irreducible closed subset has a unique generic point**. For any subset $Z \subset X$, $\mathbb{I}(Z) := \cap_{p \in Z} p$. Then $\mathbb{V}(\mathbb{I}(Z)) = \bar{Z}$ and $\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}$. Then Z is an irreducible closed subset $\Leftrightarrow \mathbb{I}(Z) = p$ is a prime ideal. Moreover, $Z = \overline{\{p\}}$, which means p is the generic point of Z.

Example: $Spec\ \mathbb{Z} := \{(p) | p\ prime\} \cup \{(0)\}$.And (0) is the generic point of $\mathbb{Z}$.

Exercise: $Spec\ \mathbb{Z}[x] = \{(0)\} \cup \{(f) | f\ prime\} \cup \{(p, f) | p\ prime\ in\ \mathbb{Z}, f\ mod\ p\ prime\ in\ \mathbb{Z}/p\mathbb{Z}[x]\}$ and draw it.

Definition: Let $X = Spec\ R$. The structure sheaf $\mathcal{O}_X$ on X is the following sheaf of rings, for open subset $U \subset X$,

$$\Gamma(U, \mathcal{O}_X) = \{\varphi : U \to \sqcup_{p \in Spec\ R} R_p | \exists U = \cup_\alpha D(g_\alpha)\ and\ s_\alpha \in R_{g_\alpha},$$

$$\varphi(Q) = image\ of\ s_\alpha\ in\ R_Q\ for\ all\ Q \in X_{g_\alpha}\}$$

Proposition: Let $X = Spec\ R$ and for any $g \in R$, $R_g = \Gamma(X_g, \mathcal{O}_X)$.

Definition: A **presheaf** $(X, \mathcal{O}_X)$ is a topological space X with a sheaf of rings on it such that $X = \cup_\alpha U_\alpha$ where $\{U_\alpha | \alpha\}$ is an open cover of X with

$$(U_\alpha, \mathcal{O}_X|_{U_\alpha}) \cong (Spec\ R_\alpha, \mathcal{O}_{Spec\ R_\alpha})$$

for some ring R_α.

Definition: For any $x \in X$, the **stalk** of the $\mathcal{O}_X$ at x is $\mathcal{O}_{X,x} := \varinjlim_{x \in U} \Gamma(U, \mathcal{O}_X)$. When $X = Spec\ R$ and $p \in Spec\ R$, $\mathcal{O}_{X,x} = R_p$.