Recap for AG 11.09

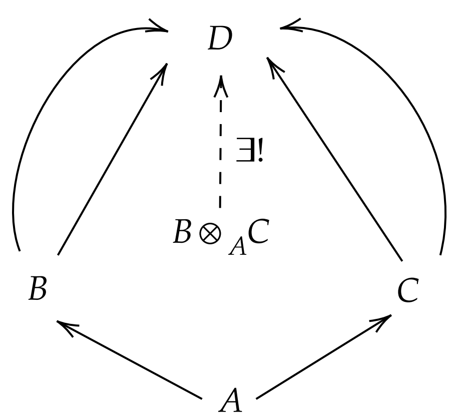
Definition 1 *Let R be a fixed ring, a prescheme over R is said to be of finite type over R (or $SpecR$) if it is quasi-compact and for any affine subset $U := SpecS \subseteq X$ we have a map $R \to S$ that is finite type.*
(Without quasi-compact we get locally finite-type)

Lemma 1 *X over R is finite type if and only if $X = U_1 \cup \ldots \cup U_n$ is a union of affine schemes and $U_i = SpecS_i$ with maps $R \to S_i$ that is finite type.*

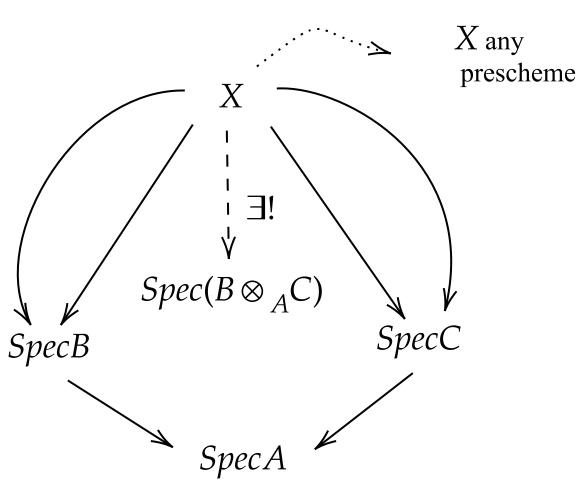
Definition 2 *We say X over k is defined over $k_0 \subseteq k$ if all of the coefficients of generators of I_i can be taken inside k_0.*

Next we define fiber product of schemes.

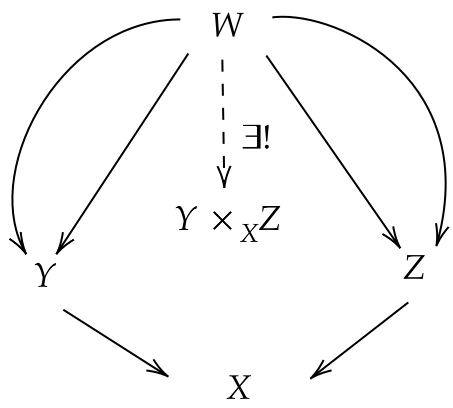
Remark/Definition/Theorem: *Let B and C be $A-$ algebras with homomorphisms $A \to B$ and $A \to C$, then for any $A-$ algebra D there exists a unique homomorphism such that the following diagram commutes (the pushout squares):*



Reversing the arrows we get the fiber product of affine schemes:



We glue the fiber products of affine schemes and get the fiber product of schemes:



Alternative definition: *Let X be finite type over k, we say X is "defined over" $k_0 \subseteq k$ if there exists X_0 over k_0 such that* $X \cong X_0 \times_{speck_0} Speck$.

We have the correspondence:

$$\{Old\ irreducible\ prevariety\ over\ k\} \cong \{Preschemes\ of\ finite\ type\ over\ k\ that\ are\ irreducible\}$$

Definition 3 *A prescheme X is reduced if it satisfies the equivalent conditions:*

1. $\forall x \in X$, $\mathcal{O}_{X,x}$ is reduced;

2. $\forall U \subseteq X$ open affine, $U = SpecR$ where R is reduced;

3. There exists open cover $X = \bigcup U_\alpha$, $U_\alpha = SpecR_\alpha$ where R_α is reduced.

Exercise: Show the above conditions are equivalent.

Let $k = \overline{k}$. Observation: Let X be defined over k_0 (so then $X = X_0 \times_{k_0} k$), define a map: $X \xrightarrow{\pi} X_0$, then π closed and open and fibers are galois orbits of k/k_0.

Example: Let $X = Spec\mathbb{C}[x]/(x^2+1)$, $X_0 = Spec\mathbb{R}(x)/\left(x^2+1\right)$, $X_0 \otimes_{\mathbb{R}} \mathbb{C} = X$, then we have map $X \to X_0$ where two points $i, -i$ of X sent to one point of X_0.

Definition 4

1. $\pi : Y \to X$ is seperated if and only if $Im\left(Y \xrightarrow{\triangle} Y \times_X X\right)$ is closed.

2. $\pi : Y \to X$ is proper if and only if it is seperated + finite type + universally closed, i.e. for all prescheme $Z \to X$ the induced map $Y \times_X Z \to Z$ is always closed.