MATH 552 Algebraic Geometry I

Oct 12 Recap

The Grassmannian $G(r,n)$ is the set of r-dimensional subspaces of the k-vector space k^n, where k is an algebraic closed field and $char\ k \neq 2$. We write $Gr(r,V)$ for the set of k-dimensional subspaces of an n-dimensional k-vector space V. For example, as sets, $Gr(1,n) = \mathbb{P}_k^{n-1}$. If V is n-dimensional k-vector space, then $Gr(r,n) \cong Gr(n-r,n)$ as sets, where the isomorphism is induced by the noncanonical isomorphism $V \cong V^*$. The first interesting Grassmannian, which is not a point or a projective space, is $Gr(2,4)$.

Definition 1. *Let V be an n-dimensional k-vector space, and fix an integer r, where $0 < r < n$. The Plücker map $\Psi : Gr(r,V) \to \mathbb{P}(\Lambda^r V)$ is defined as the following:*

$$span(v_1, \ldots, v_r) \mapsto [v_1 \wedge \ldots v_r]$$

If $(w_i = \sum_j a_{ij} v_j)_{1 \leq i \leq r}$ is another ordered basis for $\Lambda = span(v_1, \ldots, v_r)$, where $A = (a_{ij})$ is an invertible matrix, then $w_1 \wedge \cdots \wedge w_r = \det(A) v_1 \wedge \cdots \wedge v_r$. Thus, the Plücker map is a well defined map from $Gr(r,V)$ to $\mathbb{P}(\Lambda^r V)$.
Next we will show
· Ψ is injective.
· $\Psi(Gr(r,V)) \subset \mathbb{P}(\Lambda^r V)$ is closed.

Definition 2. *Given $x \in \Lambda^r V$, we say that x is a pure wedge if $x = v_1 \wedge \cdots \wedge v_r$ for $v_1, \ldots, v_r \in V$, or equivalently, if $[x]$ is in the image of the Plücker map.*

To show the Plücker map is an injection, we must describe how to recover $span(v_1, \ldots, v_r)$ for $x = v_1 \wedge \cdots \wedge v_r$.

Lemma 1. *For any non-zero $x \in \Lambda^r V$, define a k-linear map as the following:*

$$\varphi_x : V \to \Lambda^{r+1} V$$

$$v \mapsto v \wedge x$$

(1) $\dim \ker(\varphi_x) \leq r$
(2) $\dim \ker(\varphi_x) = r$ *if and only if x is a pure wedge.*
(3) If $x = v_1 \wedge \cdots \wedge v_r$, then

$$\ker \varphi_x = span(v_1, \ldots, v_r).$$

Notice that (3) implies Ψ is injective.

Theorem 1. *$\Psi(Gr(r,V))$ is closed in $\mathbb{P}(\Lambda^r V)$.*

· Cover $Gr(r,n)$ by $\mathbb{A}^{r(n-r)}$(Sketch)
Given an $r \times n$ matrix $B = (b_{ij})$ of rank r, the row space of B maps under the Plücker map to

$$(b_{11}e_1 + \cdots + b_{1n}e_n) \wedge \cdots \wedge (b_{r1}e_1 + \cdots + b_{rn}e_n) = \sum_{|J|=r} a_J e_J,$$

where the a_J are the usual Plücker coordinates. In this product, only the terms $b_{ij}e_j$ with $j \in J$ contribute to the term $a_J e_J$ and

$$(b_{1j_1}e_1+\cdots+b_{1j_r}e_{j_r})\wedge\cdots\wedge(b_{rj_1}e_1+\cdots+b_{rj_r}e_{j_r}) = (\det(b_{ij_l})_{1\leq i,l\leq r})(e_{j_1}\wedge\cdots\wedge e_{j_r}),$$

i.e., the Plücker coodinate a_J is the $r \times r$ minor of the matrix B obtained by taking all r rows and the r columns with indices in J.
We wish to describe the open subsets of $Gr(r,n)$ by $a_J \neq 0$, since $\Psi(Gr(r,n))$ can be covered by all these open subsets $\{a_J \neq 0\}$. For simplicity of notation, we will consider the case $a_{1,\ldots,r} \neq 0$, since every other case is equivalent to this one by permuting our basis for V. The corresponding minor of B is just the determinant of the leftmost $r \times r$ submatrix, and the condition $a_{1,\ldots,r} \neq 0$ means this submatrix is invertible. Multiplying on the left by the inverse of this matrix, we can replace our matrix with a new matrix B of the form

$$B = \begin{pmatrix} 1 & 0 & \ldots & 0 & b_{1,r+1} & b_{1,r+2} & \ldots & b_{1,n} \\ 0 & 1 & \ldots & 0 & b_{2,r+1} & b_{2,r+2} & \ldots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & b_{r,r+1} & b_{r,r+2} & \ldots & b_{r,n} \end{pmatrix}$$

with the same row space. Moreover, any two distinct matrices of this form will have different row spaces. Thus, we can think of these b_{ij} for $1 \leq i \leq r$ and $r+1 \leq j \leq n$ as "local coordinates" on $Gr(r,n)$, and the Plücker gives us a bijection between $\mathbb{A}^{r(n-r)}$ (with coodinates b_{ij}) and the open subset of $a_{1,\ldots,r} \neq 0$ of the Plücker image.