MATH 552 Algebraic Geometry I

Oct 8 Recap

Exercise 1.
Let $F, G \in k[u, v]$ be homogeneous polynomials of degree m and n respectively.
Let $\mathrm{Sym}^{m+n-1}(k^2)^*$ be homogeneous polynomials in u, v of degree $m+n-1$.
(a). Show that F and G have a common factor if and only if the vector subspaces $V_F, V_G \subseteq \mathrm{Sym}^{m+n-1}(k^2)^*$ containing all polynomials divisible by F and G respectively meet nontrivially.
(b). Show that F and G have a common factor if and only if $u^{n-1}F, u^{n-2}vF, ..., v^{n-1}F, u^{m-1}G, u^{m-2}vG, ..., v^{m-1}G$ are linearly dependent.
(c). Show that F and G have a common factor if and only if the determinant of a certain $(m+n)\times(m+n)$ matrix formed with coefficients of F and G is zero. (This matrix is called the *resultant* of F and G.)
(d). Assume that k is algebraically closed. Let $\mathbb{P}(\mathrm{Sym}^m(k^2)^*)$ and $\mathbb{P}(\mathrm{Sym}^n(k^2)^*)$ be the parameter spaces of m and n points on $\mathbb{P}^1$ respectively. Consider the subset $\Gamma \subseteq \mathbb{P}(\mathrm{Sym}^m(k^2)^*) \times \mathbb{P}(\mathrm{Sym}^n(k^2)^*)$ of pairs of sets of points that have a point in common. Prove that Γ is a nonempty Zariski closed set.
(e). (Challenge!) How does one generalize to more than 2 variables?

Definition.
A morphism of varieties $\pi : Y \to X$ is said to be
- *quasi-finite* if the fibers are all finite sets.
- *finite* (resp. *integral/affine*) if there is a covering $X = \bigcup_{i=1}^N U_i$ of X by open affine sets $U_i \subseteq X$ such that every $\pi^{-1}(U_i) =: V_i \subseteq Y$ is an open affine sets and every $k[U_i] \to k[V_i]$ is finite (resp. integral/affine).

Example. (Projections restricted to projective subvarieties are quasi-finite.)
Let $\Gamma, W \subseteq \mathbb{P}^n$ be some disjoint linear subvarieties with $\dim \Gamma + \dim W = n-1$. Let $X \subseteq \mathbb{P}^n$ be some projectiive variety with $X \cap \Gamma = \varnothing$. It is claimed (by Kevin) that the restriction of the projection map $\pi_{\Gamma,W}$ (defined in the Exercise on Oct 1) to X is obviously quasi-finite.
To see this, we first reduce to the case where Γ is a point. Then the fiber of a point $w \in W$ under $\pi_{\Gamma,W}$ is the intersection of X and the line L through w and Γ, which is closed in L. Note that the fiber cannot be the whole L since $\Gamma \cap X = \varnothing$. Thus, it can only be a finite set.
Proposition.
$\pi_{\Gamma,W}|_X$ is even finite.
Sketch of proof. Again, reduce to the case where Γ is a point: $\pi : \mathbb{P}^n \supseteq X \to \mathbb{P}^{n-1}$, $[x_0 : \cdots : x_n] \mapsto [x_0 : \cdots : x_{n-1}]$. Then the point $[0 : \cdots : 0 : 1]$ is not contained in X, so $f \in \mathbb{I}(X)$ of degree d is equal to $x_n^d +$ (other monomials in degree d), which gives integral equations on standard patches.

Example. (Other favorite finite maps given by Noether normalization.)
Let X be an affine variety. By Noether normalization lemma, there exist $x_1, ..., x_n \in k[X]$ such that the inclusion map $k[x_1, ..., x_n] \subseteq k[X]$ is finite (i.e. $k[X]$ is module-finite over $k[x_1, ..., x_n]$). Then this induces a finite morphism $\pi : X \to \mathbb{A}^n_k$

Exercise 2.
Show that finite maps are closed and quasi-finite.