Recap - October 15, 2018

Grassmannian $G(k,n) = \{$set of $k-$ dimensional linear subspaces of $k^{\oplus n}\}$.
$\mathbb{G}(k,n) = \{$set of $k-$ dimensional linear subspaces of $\mathbb{P}^n\} = G(k+1, n+1)$.

Example. $G(2,4) = \mathbb{G}(1,3) = \{$*lines in* $\mathbb{P}^3\}$.

$G(k,n) = \{n \times k$ matrices of rank k up to elementary column operations $\}$.
$G(k,n)$ can be realized as an algebraic variety of dimension $k(n-k)$.

Definition. *The Plücker embedding is the map*

$$G(r,n) \to \mathbb{P}(\wedge^r k^{\oplus n})$$

$$(\omega_1, ..., \omega_r) \to (\omega_1 \wedge ... \wedge \omega_r)$$

Exercise. *Suppose* $char(k) \neq 2$, V *is a vector space over* k. *Show directly that a vector* $\omega \in \wedge^2 V$ *is a pure wedge iff* $\omega \wedge \omega = 0$. *Conclude that* $G(2,V)$ *is a variety cut out by quadrics (under Plücker embedding). Observe in particular that* $G(2,4)$ *is the non-degenerate quadric hypersurface in* $\mathbb{P}^5$.

Exercise. *Fix an irreducible conic* C *in* $\mathbb{P}^2$. *Show that the set of lines in* $\mathbb{P}^2$ *that fail to meet* C *in exactly two points is a closed subset of the Grassmannian.*

Theorem (Krulls Hauptidealsatz)**.**
(Geometric version): Suppose $k = \bar{k}$, $X \subset \mathbb{P}^n_k$ *is a (quasi)-projective variety with dimension* d *and* $f \in k[x_0, ..., x_n]$ *is homogeneous,* $f \notin \mathcal{I}(X)$. *Then every component of* $\mathcal{V}(f) \cap X$ *has dimension* $d-1$.
(Algebraic version): Suppose R *is a Noetherian ring and* $f \in R$. *Then every minimal prime* $\mathfrak{p}$ *containing* f *has height at most* 1.