MATH 552 ALGEBRAIC GEOMETRY I

OCTOBER 17 RECAP

Exercise (Tautological line bundle) Consider

$$L = \{(l, x) | l \in \mathbb{P}^n, x \in k^{n+1}, x \in l\} \subseteq \mathbb{P}^n \times k^{n+1} \text{ (or } \mathbb{P}(V) \times V)$$

(1) Show this is a closed algebraic subset by finding explicit defining equations.

(2) Consider the natural map $\pi : L \to k^{n+1}$ given by projection onto the second factor. Describe the fibers. Give equations that cut it out. Can you identify it?

(3)Consider $\eta : L \to \mathbb{P}^n$ given by projection onto the first factor. Describe the fibers. Give equations that cut it out. Justify why this is called a line bundle.

Comments on Krull's PID Theorem

- If X is some variety of dimension d and $f \in \Gamma(X, \mathcal{O}_X)$ is nonconstant, then every component of $\mathbb{V}(f) \subseteq X$ has codimension 1 in X.
- If $X \subseteq \mathbb{P}^N$ is a variety of dimension n and $f \in k[x_0, \ldots, x_N]$ is homogeneous, nonzero, and $f \notin \mathbb{I}(X)$, then $\mathbb{V}(f) \cap X \subseteq X$ has codimension 1.

Corollary If $X \subseteq \mathbb{P}^n$ is a variety of codimension c, then there exists homogeneous polynomials $f_1, \ldots, f_c$ such that X is a component of $\mathbb{V}(f_1, \ldots, f_c) \subseteq \mathbb{P}^n$. (N.B. all components have codimension less than or equal to c)

The proof uses:

Prime Avoidance Let R be a ring and $I, P_1, \ldots, P_m$ be ideals of R such that $P_3, \ldots, P_m$ are prime. If $I \subseteq P_1 \cup \cdots \cup P_m$, then there exists an i such that $I \subseteq P_i$.

Exercise Prove Prime Avoidance in the limited case where all of the P_i are prime.

Theorems on Dimension of Fibers (Memorize this!) Suppose that $\pi : Y \to X$ is a dominant morphism of varieties with $dim(X) = d$.

(1) The dimension of any component of any fiber of π is at least $dim(Y) - dim(X)$.

(2) There exists a nonempty open subset of X such that all components of all fibers have dimension exactly $dim(Y) - dim(X)$.

(3) The function s which sends $y \mapsto dim(\pi^{-1}(\pi(y)))$ is upper-semi-continuous on Y (i.e. $\forall \alpha \in \mathbb{R}, \{y \in Y | s(y) > \alpha\}$ is closed in Y). (jumps on closed sets)