Algebraic Geometry I. Recap 10/23

Thm. Let $\pi : Y \to X$ regular, $W \subseteq X$ irreducible, $Z = \pi^{-1}(W)$, with $\pi^{-1}(\{w\})$ irreducible of constant dimension r for all $w \in W$.

Then, Z is irreducible.

Degree.

Thm. Let $X \subseteq \mathbb{P}^n$ projective variety, not a hypersurface (so $\dim X < n-1$)

Then, for a general point $\alpha \in \mathbb{P}^n \setminus X$, $\alpha \notin$ any secant line to X, and projection from α gives

$$\pi : X \to \mathbb{P}^{n-1} \text{ finite \& birational onto its image}$$

Def. $\pi : Y \to X$ is birational if it gives an isomorphism over a nonempty open subset of X.

i.e., $\exists U \subseteq X$ open st $\pi^{-1}(U) \xrightarrow{\sim} U$.

"π has a rational inverse"

Lemma. Let $\pi : Y \to X$ dominant. π is birational

$$\iff$$

the induced map $k(X) \subseteq k(Y)$ is an isomorphism.

exercise. Fix char $k = 0$. Let $X \subseteq \mathbb{P}^n_k$. Show that X is exercise. Fix char $k=0$. Let $X \subseteq \mathbb{P}^n_k$. Show that X is birational to some hypersurface of $\mathbb{P}^m$, some m.

Hint: $Y = \mathbb{V}(f) \subseteq \mathbb{A}^n$

$$k(Y) = \mathrm{Frac}\left(\frac{k[x_1, \dots, x_n]}{(f)}\right)$$

assume f is monic in some variable. (Noether normalization)

so $$k(Y) = \frac{k(x_1, \dots, x_{n-1})[x_n]}{(f)}$$

2nd pf of this fact assuming the theorem:

Let $X \subseteq \mathbb{P}^n$ not a hypersurface, say of dimension d.

Then there is $X_1 \subseteq \mathbb{P}^{n-1}$ and $X \xrightarrow{\pi} X_1$ birational map.

Repeat:

$$\begin{array}{ccccccc} X & \to & X_1 & \to & \cdots & \to & X_{n-d-1} \\ \cap\!\mid & & \cap\!\mid & & & & \cap\!\mid \\ \mathbb{P}^n & & \mathbb{P}^{n-1} & & & & \mathbb{P}^{d+1} \end{array}$$

codim 1, hence hypersurface