MATH 552 Algebraic Geometry I

Oct 26 Recap

Definition. (Fano variety $F_r(X)$.)
Let $X \subset \mathbb{P}^n$ be a projective variety. Define the *Fano variety of* r*-planes in* X by $F_r(X) := \{\Lambda \in \mathbb{G}(r,n) \mid \Lambda \subseteq X\}$.

Lemma.
$F_r(X)$ is closed in $\mathbb{G}(r,n)$.
Sketch of proof.
We can reduce to the case where $X = \mathbb{V}(f)$ is a hypersurface, since $F_r(X) = \bigcap_{f \in \mathbb{I}(X) \text{ homogeneous}} F_r(\mathbb{V}(f))$. Also, it suffices to check the closedness on standard affine patches of $\mathbb{G}(r,n)$. WLOG we can consider the affine patch containing elements that can be represented by matrices of the form

$$\Lambda = \begin{pmatrix} 1 & \cdots & 0 \\ 0 & & 1 \\ * & \cdots & * \\ * & \cdots & * \end{pmatrix}.$$

Denote the i-th column of this matrix by x_i ($i = 0, ..., r$). We have $\Lambda \subseteq \mathbb{V}(f)$ if and only if $f(\lambda_0 x_0 + \cdots \lambda_r x_r) = 0$ for all $\lambda_i \in k$. Treat λ_i's as variables and expand this equation. The coefficients gives polynomials in x_i's. Note that $f(\lambda_0 x_0 + \cdots \lambda_r x_r) = 0$ for all $\lambda_i \in k$ if and only if all of these polynomials vanish. Thus, $F_r(X)$ is the vanishing set of these polynomials and hence is closed in this affine patch.

Definition. ($I(r;d,n)$.)
Let $\mathbb{P}^N$ represent the collection of all hypersurfaces of degree d on $\mathbb{P}^n$, and recall that $\mathbb{G}(r,n)$ is the collection of all r-planes in $\mathbb{P}^n$. Define $I(r;d,n) := \{(X,\Lambda) \in \mathbb{P}^N \times \mathbb{G}(r,n) \mid \Lambda \subseteq X\}$.
Remark.
Note that $N = \binom{n+d}{d} - 1$ and that $\dim \mathbb{G}(r,n) = (r+1)(n-r)$.

Proposition.
$I(r;d,n)$ is closed in $\mathbb{P}^N \times \mathbb{G}(r,n)$, irreducible, and of dimension $(r+1)(n-r) + \binom{n+d}{d} - \binom{r+d}{d} - 1$.
Proof.
The proof for closedness is the same as the proof of the lemma, except we got polynomials in x_i's and coefficients of f instead. To prove the remaining results, consider the projection $\pi : I(r;d,n) \to \mathbb{G}(r,n)$. We claim that π is surjective and that all fibers of π are linear subspaces of dimension $\binom{n+d}{d} - \binom{r+d}{d} - 1$. This would imply that $I(r;d,n)$ is irreducible, and that its dimension is $(r+1)(n-r) + \binom{n+d}{d} - \binom{r+d}{d} - 1$. With the power of change of coordinates in $\mathbb{P}^n$, it suffices to consider the fiber of $\Lambda = \mathbb{V}(x_{r+1}, ..., x_n) \in \mathbb{G}(r,n)$ is nonempty and has the correct dimension. Observe that $\pi^{-1}(\Lambda)$ contains all hypersurfaces $X \in \mathbb{P}^N$ containing Λ. Note that every hypersurface $X = \mathbb{V}(f)$ is given by a homogeneous $f \in k[x_0, ..., x_n]$ of degree d. We have $\Lambda \subseteq \mathbb{V}(f)$ if and only if $f(x_0, ..., x_r, 0, ..., 0) = 0$. This is equivalent to f lying in the kernel of the surjection from the vector space $\mathrm{Sym}^d(k^{n+1})^*$ of homogeneous polynomials in $x_0, ..., x_n$ to the vector space $\mathrm{Sym}^d(k^{r+1})^*$ of homogeneous polynomials in $x_0, ..., x_r$. Then, the fiber $\pi^{-1}(\Lambda)$ is the projectivization of this kernel and hence has dimension $\binom{n+d}{d} - \binom{r+d}{d} - 1$, as desired.

Corollary. Let $\mathbb{V}(f) \subset \mathbb{P}^n$ be a hypersurface of degree d, i.e. f has degree d. Then, the "expected dimension" of $F_r(\mathbb{V}(f))$ is $(r+1)(n-r) - \binom{r+d}{d}$. This is the dimension of any component of $F_r(\mathbb{V}(f))$ for "general f" if and only if every $\mathbb{V}(f)$ contains some $\Lambda \in \mathbb{G}(r, n)$.
Proof.
Analyze the projection $\pi : I(r; d, n) \to \mathbb{P}^N$.

Examples.
When $(r; n, d) = (1, 3, 3)$, we have $\dim F_r\big(\mathbb{V}(f)\big) = 2$.
When $(r; n, d) = (1, 3, 2)$, we have $\dim F_r\big(\mathbb{V}(f)\big) = 1$.
For $(r; n, d) = (1, 3, 3)$, we have ...

Theorem.
Every cubic surface in $\mathbb{P}^3$ contains a line. The general one contains finitely many of them.
Proof.
By "dimension of fibers" theorems, it suffices to find a single cubic surfaces with only finitely many lines on it. Hence, consider $X := \mathbb{V}(x_0^3 - x_1x_2x_3) \subseteq \mathbb{P}^3$. Note that no lines on X can intersect $X \cap \mathbb{A}_{x_0 \neq 0} = \mathbb{V}(x_1x_2x_3 - 1) \subseteq \mathbb{A}^3$. Indeed, every line in $\mathbb{A}^3$ is given by $(a_1, a_2, a_3) + t(b_1, b_2, b_3)$ $(t \in k)$ for some $a_1, a_2, a_3, b_1, b_2, b_3 \in k \backslash \{0\}$, but $(a_1+b_1t)(a_2+b_2t)(a_3+b_3t) \neq 1$. Thus, every line on X must be contained in $\mathbb{V}(x_0) \cap \mathbb{V}(x_0^3 - x_1x_2x_3) = \mathbb{V}(x_0, x_1x_2x_3) = \mathbb{V}(x_0, x_1) \cup \mathbb{V}(x_0, x_2) \cup \mathbb{V}(x_0, x_3)$, which is the union of three lines.