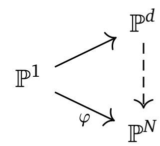
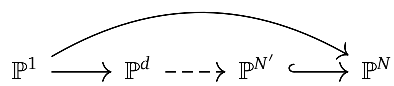
Math 552 Algebraic Geometry I

Proposition. *Let $\mathbb{P}^1 \xrightarrow{\varphi} \mathbb{P}^N$ be a nondegenerate (that is, the image of φ is not contained in a hyperplane) map of varieties. Then it factors as*



Where $\mathbb{P}^1 \to \mathbb{P}^d$ is the rational normal curve of degree d and $\mathbb{P}^d \dashrightarrow \mathbb{P}^N$ is a projection away from a linear subvariety disjoint from a rational normal curve.

Remark. If φ is degenerate, we instead get



Where $\mathbb{P}^1 \to \mathbb{P}^d$ is the rational normal curve of degree d and $N > N'$.

Definition. A rational map $\pi : X \dashrightarrow Y$ is "a regular function $U \xrightarrow{\pi} Y$ on on open subset $\varnothing \neq U \subset X$." More formally, $\pi : X \dashrightarrow Y$ is an equivalence class of pairs $(U \subseteq X \text{ open}, \pi : U \to Y)$ with $(U_1, \pi_1) \sim (U_2, \pi_2)$ if and only if $\pi_1\big|_{U_1 \cap U_2} = \pi_2\big|_{U_1 \cap U_2}$.

Observe that for $\varnothing \neq U_1, U_2 \subseteq X$, given two morphisms $U_1 \xrightarrow{\pi_1} Y$ and $U_2 \xrightarrow{\pi_2} Y$ with $\pi_1\big|_{U_1 \cap U_2} = \pi_2\big|_{U_1 \cap U_2}$, we can glue them to get a regular morphism $\pi : U_1 \cap U_2 \to Y$ with $\pi\big|_{U_1} = \pi_1$ and $\pi\big|_{U_2} = \pi_2$. So, any rational map $\pi : X \dashrightarrow Y$ has a largest set on which it is well-defined.

Example. Let $\Gamma \subseteq \mathbb{P}(V)$ be a linear subspace, and let $\pi_\Gamma : \mathbb{P}(V) \dashrightarrow \mathbb{P}(L)$ be a rational map defined on $\mathbb{P}(V) \setminus \Gamma$, where L is a linear subspace of codimension complimentary to that of Γ and disjoint from Γ.

Example. A rational map $X \dashrightarrow k$ is the same as a rational function on X (all of them give $k(X)$, the function field of X).

Definition. The *Veronese embedding* $\mathbb{P}^n \xrightarrow{\nu_d} \mathbb{P}^{\binom{n+d}{d}-1}$ is the map defined on coordinates by

$$[x_0 : \cdots : x_n] \mapsto [x_0^d : x_0^{d-1}x_1 : x_0^{d-2}x_1^2 : \cdots : x_0^{a_0} \ldots x_n^{a_n} : \ldots]$$

where $\sum a_j = n$. Note that $\binom{n+d}{d} - 1$ is the dimension of the space of homogeneous polynomials of degree d in $x_0, \ldots, x_n$.

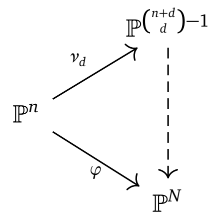
This is a closed embedding. In order to see this, we may show that the image is closed, and that the map is an isomorphism onto its image (i.e. construct an inverse), by setting $\frac{z_1}{z_0} = \frac{x_1}{x_0}$ (and so on), where the z_i are the coordinates on $\mathbb{P}^{\binom{n+d}{d}-1}$. Note that one can also show that the image of the Veronese embedding can be defined by quadrics. Without coordinates, the Veronese embedding is the map $\mathbb{P}(V) \to \mathbb{P}(\mathrm{Sym}^d(V))$.

Exercise. Let $k = \overline{k}$, and let $X \subseteq \mathbb{P}^n$.

(a) Show that X can be given as the zero locus of homogeneous polynomials of the same degree.
(b) Show that X is isomorphic to a linear section of a Veronese n-fold (that is, given $\nu_d(\mathbb{P}^n)$ for some d, $X \cong \nu_d(\mathbb{P}^n) \cap (\text{some linear space})$).
(c) Show that X is isomorphic to an intersection of quadrics (in some other projective space).

Exercise. Let $X = \mathbb{V}(f) \subseteq \mathbb{P}^n$ be a hypersurface. Show that $\mathbb{P}^n \setminus X$ is affine. (Hint: do the case where X is a hyperplane first.)

Remark. For a nondegenerate map $\mathbb{P}^n \xrightarrow{\varphi} \mathbb{P}^N$, we have a factorization



where the map $\mathbb{P}^{\binom{n+d}{d}-1} \dashrightarrow \mathbb{P}^N$ is some projection.