Summary: we proved the Hilbert-Samuel function of a variety is eventually polynomial, so it agrees with the Hilbert polynomial for large enough inputs, and the leading term contains information about the degree and dimension of the variety.

Definition 1. Let $X \subseteq \mathbb{P}^n$ be of pure codimension c. We define the degree of X (written $\deg X$) to be the number of intersections between X and a general linear space of dimension c. (General means it is part of an open set of all linear spaces that satisfy this property in $\mathbb{G}(c, n)$.)

Theorem 1. *Let $p_X(t) \in \mathbb{Q}[t]$ be the Hilbert polynomial of X, that is,*

$$p_X(t) = \frac{\deg X}{(\dim X)!} t^{\dim X} + \textit{lower order terms.}$$

Then $p_X(m) = \dim_k (S_X)_m$ for all sufficiently large $m \gg 0$, where $S_X = k[X] = k[x_0, \ldots, x_n]/I(X)$, and $(S_X)_m$ is the subspace of $\deg m$ homogeneous polynomials.

Exercise 1.

1. $X \subseteq \mathbb{P}^n$, calculate the Hilbert polynomial of the dth Veronese image $\nu_d(X)$ inside $\mathbb{P}^{\binom{n+d}{d}-1}$.
2. In particular, what is the Hilbert polynomial of the twisted cubic?
3. What about rational normal curves in general?
4. What about $\nu_d(\mathbb{P}^n)$? (Hint: there is a surjective map from the homogeneous polynomials over the target projective space to the source projective space, and the degree shifts.)

Exercise 2. Let $\Sigma_{n,m} \subseteq \mathbb{P}^N, N = (n+1)(m+1) - 1$ be the Segre image. What is the Hilbert polynomial? (Hint: first consider the homogeneous polynomials of $\deg l$ on $\mathbb{P}^N$, restrict to give all homogeneous polynomials of bidegree (l, l) on $\Sigma_{n,m} \simeq \mathbb{P}^n \times \mathbb{P}^m$.

Lemma 2.

1. *Let $P \in \mathbb{Q}[t]$ be any polynomial of degree d. Then $P(n) \in \mathbb{Z}$ for all $n \gg 0, n \in \mathbb{Z}$ (in which case we say P is eventually polynomial or numerically polynomial) if and only if $P(t)$ is a binomial polynomial:*

$$P(t) = a_d \binom{t+d}{d} + a_{d-1} \binom{t+d-1}{d-1} + \cdots + a_0,$$

with $a_i \in \mathbb{Z}$. We define

$$\binom{t+l}{l} := \frac{(t+l)\cdots(t+1)}{l!} \in \mathbb{Q}[t].$$

2. *Let $f : \mathbb{Z} \to \mathbb{Z}$. Let $\Delta f : \mathbb{Z} \to \mathbb{Z}$ be the difference polynomial $\Delta f(n) = f(n) - f(n-1)$. Then Δf is eventually polynomial of degree $d-1$ if and only if f is eventually polynomial of degree d.*

Lemma 3. *Let M be a finitely generated graded module over $k[x_0, \ldots, x_n]$. Then $l \mapsto \dim_k M_l$ is eventually polynomial.*

Corollary 3.1. *The Hilbert-Samuel function is eventually polynomial.*