Summary: we defined what a complete variety is and proved that projective varieties are complete.

Theorem 1. *Let $X \subseteq \mathbb{P}^n$ be a projective variety, and $\pi : X \to Y$ be any map of varieties. Then $\pi(X)$ is closed in Y. (Note that we only need Y to be separated, which is satisfied by a variety.)*

Theorem 2. *Let $X \subseteq \mathbb{P}^n$ be a projective variety, Y any variety. Then $\pi_Y : X \times Y \to Y$ is a closed mapping.*

Corollary 2.1. *Let $k = \overline{k}$. Let $X \subseteq \mathbb{P}^n$ be a projective variety. We have $\Gamma(X, \mathcal{O}_X) = k$. (That is, the only global regular functions are constant functions.)*

Definition 1. A variety X is said to be complete if for any other variety Y, the projection map onto the second factor $\pi_Y : X \times Y \to Y$ is a closed mapping.

Corollary 2.2. *Let X be a complete variety. If $\pi : X \to k^n$ is a regular map, then $\pi(X) \simeq \{*\}$ (the image is a point).*

Theorem 3. *Projective varieties are complete.*

Remark 1.

- Complete varieties in algebraic geometry are like compact manifolds in differential geometry. However there are complete varieties that are not projective, even though all projective varieties are compact.
- The proof of this theorem involves reducing to the affine case, then applying elimination theory.