MATH 552 Algebraic Geometry I

Sept 10 Recap

1 Products (Mostly a preview)

Definition. The *product* of two affine algebraic sets $X \subseteq k^n$ and $Y \subseteq k^m$ is defined as the closed algebraic set $X \times Y$ in k^{n+m}

Remark. The Zariski topology on $X \times Y$ is NOT the product topology of the Zariski topologies of X and Y.

Exercise 1.

(a). Show that the Zariski topology on k^2 is not the product topology on $k \times k$.

(b). With notion of products of affine algebraic sets on the left, show that $k[X \times Y] \cong k[X] \otimes_k k[Y]$ and deduce that $X \times Y$ is a categorical product.

(c). If X and Y are disjoint affine algebraic sets in k^n, show that $k[X \cup Y] \cong k[X] \times k[Y]$.

Definition. Let $\{U_i \cong k^n\}_{i=0}^n$ be the standard affine open cover of $\mathbb{P}_k^n$. Let $\{V_j \cong k^m\}_{i=0}^m$ be the standard affine open cover of $\mathbb{P}_k^m$. View $\mathbb{P}^n \times \mathbb{P}^m := \bigcup_{i,j} U_i \times V_j$ as glueing of topological spaces to give Zariski topology on $\mathbb{P}_k^n \times \mathbb{P}_k^m$

Remark. To see that this definition is categorical, we still need the notion of morphisms between projective varieties.

Lemma.

(a). If $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ are projective algebraic sets, then the subset $X \times Y \subseteq \mathbb{P}^n \times \mathbb{P}^m$ is Zariski closed.

(b). A subset $Z \subseteq \mathbb{P}^n \times \mathbb{P}^m$ is Zariski closed if and only if Z is the common zeros of some *bi-homogeneous* polynomials, i.e. polynomials $f \in [x_0, ..., x_n, y_0, ..., y_m]$ with some positive integers d, e such that $f(\lambda \underline{x}, \mu \underline{y}) = \lambda^d \mu^e f(\underline{x}, \underline{y})$ for all $\underline{x} \in k^n$, $\underline{y} \in k^m$, and $\lambda, \mu \in k$.

2 $\text{Spec}(R)$ as a topological space

Definition. Let R be a commutative ring. Define $\text{Spec}(R)$ to be the set of prime ideals of R.
The *Zariski topology* of $\text{Spec}(R)$ is defined by declaring the closed sets to be subsets of the form $\mathbb{V}(\Sigma) := \{P \in \text{Spec}(R) \mid P \supset \Sigma\}$ for some $\Sigma \subseteq R$.
Given $X \subseteq \text{Spec}(R)$, define $\mathbb{I}(X) := \bigcap_{P \in X} P$.

Remark. Many of same properties for algebraic sets hold for $\text{Spec}(R)$. For example, $\text{Spec}(R)$ are also quasicompact. We also have the similar result $\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}$ to the Nullstellensatz, but it is much easier to prove in this setting.

Exercise 2.
(a). Show that every irreducible closed subset of $\text{Spec}(R)$ is the closure of a uniquely determined point, which is in fact $\mathbb{I}(X)$ and called the *generic point*.
(b). Let $\varphi : R \to S$ be a ring homomorphism. Show that it induces a continuous map $\text{Spec}(R) \leftarrow \text{Spec}(S)$ defined by $Q \cap R = \varphi^{-1}(Q) \leftarrow\!\shortmid Q$.
Remark. The induced map is well-defined since $Q \cap R = \varphi^{-1}(Q)$ must be a prime ideal of R. If R and S are finite-type k-algebras, we have even stronger result:

Lemma. Let R and S be finite-type k-algebras. Let $\varphi : R \to S$ be a ring homomorphism. If $Q \in \text{Spec}(S)$ is a maximal ideal, then so is $Q \cap R = \varphi^{-1}(Q)$.
Remark. For homomorphisms of general commutative rings, the preimage of a maximal ideal may not be maximal. For instance, consider the inclusion $\mathbb{Z} \to \mathbb{Q}$. The only maximal ideal of $\mathbb{Q}$ is (0), but its preimage, which is $(0) \subseteq \mathbb{Z}$, is not a maximal ideal.

Exercise 3.
(a). Let $\pi : Y \to X$ be a continuous map of topological spaces. If $Z \subseteq Y$ is irreducible, then so is its image $\pi(Z) \subseteq X$.
(b). Let X be a topological space. If $Z \subseteq X$ is irreducible, then so is its closure $\overline{Z}$.
(c). Let $\pi : Y \to X$ be a regular map of affine algebraic sets. Show that $\pi(Y)$ is dense in X (i.e. π is *dominant*) if and only if $k[X] \to k[Y]$ is injective.