MATH 552 Algebraic Geometry I

Sep 12 Recap

Definition 1. *Let X be a topological space, the local dimension $\dim_x X$ at closed point $x \in X$ is the maximal length of a chain of irreducible closed subsets of X starting with $\{x\}$*

$$x = Z_0 \subsetneq Z_1 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n \subset X.$$

And the dimension of X is $\dim X = \sup_{x \in X, x \text{ closed}} \dim_x X$.

Example 1. *Let $X = k^n$, where k is a field. We have the following chain of irreducible closed subsets of k^n:*

$$\{\underline{0}\} = \mathbb{V}(x_1, \ldots, x_n) \subsetneq \mathbb{V}(x_1, \ldots, x_{n-1}) \subsetneq \cdots \subsetneq \mathbb{V}(x_1) \subsetneq \mathbb{V}(0) = X.$$

Thus, $\dim_{\underline{0}} X \geq n$.

Example 2. $X = \mathbb{V}(x) \cup \mathbb{V}(y, z) = \mathbb{V}(xy, xz) \subset k^3$. *If $x \in \mathbb{V}(y, z)$,* $\dim_x X = 2$. *If $x \in \mathbb{V}(x) \backslash \{\underline{0}\}$,* $\dim_x X = 1$. *Thus,* $\dim X = 2$.

Example 3. *For $R = k[x_1, x_2, \ldots, x_n, \ldots]$, $\dim \mathit{Spec}\ R = \infty$, since we have the following chain of irreducible closed subsets of X:*

$$\mathbb{V}(x_1, x_2, \ldots, x_n, \ldots) \subsetneq \mathbb{V}(x_1, x_2, \ldots, x_n, \ldots) \subsetneq \cdots \subsetneq \mathbb{V}(x_n, \ldots) \subsetneq \ldots.$$

But if R is Noetherian, $\dim_{m_x} \mathit{Spec}\ R < \infty$ for all maximal idelas $m_x \subset R$.

Observation: let $X \subset k^n$ be an affine algebraic set, $x \in X$ is a closed point which corresponds to a maximal ideal $m_x \subset k[X]$. Then

$$\dim_x X = \dim_{m_x} \mathit{Spec}\ k[X] = \text{maximal length of all chains of prime ideals.}$$

Theorem 1. *If k is a field, R is finite type $k-$algebra and R is a domain, then* $\dim \mathit{Spec}\ R = tr\ \deg_k \mathit{Frac}(R)$. *In particular,* $\dim k^n = \dim k[x_1, \ldots, x_n] = n$. *Moreover,* $\dim R$ *is the number of variables in any Noether normalization of R.*

Theorem 2. *(Cohen-Seidenberg Theorems/Exercises)R,S are rings and $\varphi : R \to S$ is an integral ring homomorphism.*
(1) Lying over: For all prime ideals $P \subset R$, there exist prime ideals $Q \subset S$ such that $Q \cap S = P$.
(2) Incomparability: For prime ideal $P \subset R$ and two distinct prime ideals $Q_1, Q_2 \subset S$ satisfying $Q_1 \cap R = Q_2 \cap R = P$, then $Q_1 \not\subseteq Q_2$ and $Q_2 \not\subseteq Q_1$.
(3) Going up: Let

$$P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$$

be a chain of prime ideals of R and Q_0 is a prime ideal of S which satisfies $Q_0 \cap R = P_0$, then we have a chain of prime ideals of S:

$$Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_n$$

where $Q_i \cap R = P_i$, *for each* $i = 1, \ldots, n$.
(4) Going down: (we assume that R is a normal domain) Let

$$P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$$

be a chain of prime ideals of R *and* Q_n *is a prime ideal of* S *which satisfies* $Q_n \cap R = P_n$, *then we have a chain of prime ideals of S:*

$$Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_n$$

where $Q_i \cap R = P_i$, *for each* $i = 0, \ldots, n-1$.

Exercise: Show any UFD is normal.

Corollary 1. *If* $\varphi : R \to S$ *is an integral ring homomorphism, then* $\dim R = \dim S$.

Lemma 1. $\dim k[x_1, \ldots, x_n] = n$, *where* k *is a field.*

Remark 1. *Using Lemma 1 and Corollary 1, we can prove Theorem 1.*