Recap for Algebraic Geometry

September 14, 2018

Last time: We have proved that dim $k[x_1, \ldots, x_n] = n$.

Exercise. Show that given any maximal ideal $\mathfrak{m} \subseteq k[x_1, \ldots, x_n]$ and given any saturated chain of prime ideals

$$0 = P_0 \subsetneq P_1 \subsetneq P_2 \subsetneq \ldots \subsetneq P_r = \mathfrak{m},$$

we have $r = n$.

Corollary. Let R be a finite-type k-algebra. If we take any two primes $P \subseteq Q$ in R, then any saturated chains of prime ideals

$$P = P_0 \subsetneq P_1 \subsetneq P_2 \subsetneq \ldots \subsetneq P_r = Q$$

have the same length.

Note: To prove this corollary we need to use going-down theorem all over the place.

New stuff for today: **Sheaves of Rings of functions** (gentle introduction)

Definition. Let X be some topological space. The **sheaf** of rings of functions, denoted by $\mathcal{O}$, is such that:

- For every open set $U \subseteq X$, we have a set of functions

  $$\Gamma(U, \mathcal{O}) = \mathcal{O}(U) = \{\phi : U \to \text{ something } \},$$

  where the ring $\Gamma(U, \mathcal{O})$ reads sections of $\mathcal{O}$ on U;

- For $U_1 \subseteq U_2$, there exists restriction of maps

  $$\mathcal{O}(U_2) \to \mathcal{O}(U_1)$$
  $$(\phi : U_2 \to \text{ something }) \mapsto \phi|_{U_1}$$

- $\mathcal{O}$ should be determined locally. Take any open set U and a function $\phi : U \to$ something so that when U has the open cover $U = \bigcup U_i$ and $\phi|_{U_i} = \mathcal{O}(U_i)$ for $\forall i$, we have $\phi \in \mathcal{O}(U)$.

Example 1. Let X be a smooth manifold and take $\mathcal{O}$ to be C^∞ $\mathbb{R}$-valued function. Then this is a natural sheaves on X. (We can replace smooth by analytic and $\mathbb{R}$-valued functions by $\mathbb{C}$-valued analytic functions, etc)

Example 2. Let X be a topological space. Take $\mathcal{O}$ to be locally constant $\mathbb{Z}$-valued functions. This is also a sheaves on X.

Another goal for today:

Definition. Let $k = \bar{k}$, and let $X \subseteq k^n$ be an affine algebraic set. The **sheaf of regular functions on** X is given by $\mathcal{O}_X(U) = \{\phi : U \to k\}$ where U is Zariski open, and for all points $x \in U$ there exists an open set $x \in V \subseteq U$ and $f, g \in k[x_1, \ldots, x_n]$ such that f, g never vanish on V, we have $\phi|_V = \frac{f}{g}\Big|_V$.

Theorem. Let $X \subseteq k^n$ be an affine algebraic set. Then we have

1. $\Gamma(X, \mathcal{O}_X) = k[X]$;
2. For any $g \in k[x_1, \ldots, x_n]$ with $X_g = X \setminus \mathbb{V}(g)$ we have $\Gamma(X_g, \mathcal{O}_X) = k[X][\frac{1}{g}]$.

In particular $2 \implies 1$ by setting $g = 1$.

Exercise. Prove that $k[X] \subseteq \Gamma(X, \mathcal{O}_X)$.