Summary: we proved the equality between the global sections of the sheaf of regular functions on X and the coordinate ring of X using a partition of unity. We identified the fraction field of the coordinate ring of X with the direct limit of rings of sections over all open subsets of X. Using this identification, we may realize any ring of sections of regular functions as a subring of the fraction field. Finally, we defined the stalk of the sheaf of regular functions at $x \in X$ to be the direct limit of rings of sections of regular functions over open neighbourhoods of x.

Let $k = \overline{k}$ be an algebraically closed field. Let $R = k[x_1, \ldots, x_n]$.

Theorem 1. *Let $X \subseteq k^n$ be an affine algebraic set. Then for any $f \in R$ and $X_f = X \setminus V(f)$,*

$$k[X][1/f] = \Gamma(X_f, \mathcal{O}_X).$$

In particular, when $f = 1$, we have $k[X] = \Gamma(X, \mathcal{O}_X)$.

Remark 1. On the LHS is the ring of polynomials restricted to X with f inverted; on the RHS is the sheaf of k-valued locally rational functions on X_f.

Proof. The forward containment is easy because we can just restrict the domain of the functions in the LHS to $X_f \subseteq X$, and they are already rational functions. Next we show the backward containment. Let $\varphi : X_f \to k$ be a section of $\mathcal{O}_X$ on X_f. By definition of $\mathcal{O}_X$, given any $x \in X_f$, we may find a Zariski open neighbourhood $x \in V_x \subseteq X_f$, $p_x, q_x \in R$ with $V(q_x) \cap V_x = \varnothing$ and $\varphi = p_x/q_x$ on V_x.

Since distinguished open sets of the form $V(g)$ form a basis for the Zariski topology, we may restrict V_x to a smaller distinguished open set containing x, $V(g_x)$. Without loss of generality, assume $V_x = V(g_x)$. We may restrict once again to

$$\begin{aligned} V_x &= (X \setminus V(g_x)) \cap (X \setminus V(q_x)) \\ &= X \setminus (V(g_x) \cup V(q_x)) \\ &= X \setminus V(g_x q_x) \\ &= X_{g_x q_x}. \end{aligned}$$

Replace p_x/q_x by $g_x p_x / g_x q_x$, which is equivalent. In particular, now V_x, p_x, q_x are chosen such that $V_x = X_{q_x}$. This is good because now we can say the X_{q_x}'s cover X_f. (Another way of doing this: since q_x doesn't vanish on V_x, we have $V_x \subseteq X_{q_x}$ hence $V(q_x) \subseteq V(g_x)$. Apply Nullstellensatz.)

Assume X is irreducible. Then for any $x_1, x_2 \in X$, $V_{x_1} \cap V_{x_2} \neq \varnothing$. On the intersection, we have

$$\begin{aligned} \frac{p_{x_1}}{q_{x_1}} &= \frac{p_{x_2}}{q_{x_2}} \\ \Rightarrow p_{x_1} q_{x_2} &= p_{x_2} q_{x_1}. \end{aligned}$$

Consider $Y_1 = V(p_{x_1} q_{x_2} - p_{x_2} q_{x_1}) \supset V_{x_1} \cap V_{x_2} = X \setminus Y_2$. Then $X = Y_1 \cup Y_2$ is irreducible and $Y_2 \neq X$ implies $Y_1 = X$. Hence the equality above holds on all of X.

Since X_f is quasi-compact, we can find a finite subcover by $X_{q_{x_1}}, \ldots, X_{q_{x_r}}$. (This is because on the algebra side we are able to write f^s as a finite sum in terms of the q_x's.) In particular,

$$\begin{aligned} X_f &= \cup_{i=1}^r X \setminus V(q_{x_i}) \\ &= X \setminus V(q_{x_1}, \ldots, q_{x_r}) \\ &= X \setminus V(f). \end{aligned}$$

This implies $V(f) \cap X = V(q_{x_1}, \ldots, q_{x_r}) \cap X$. Then $V(f, I(X)) = V(q's, I(X))$. Nullstellensatz implies $f^s \in (q's, I(X))$ for some pwer $s \in \mathbb{N}$. That is,

$$f^s = \sum_{i=1}^{r} u_i q_{x_i} + h, u_i \in R, h \in I(X).$$

We use this equation like a partition of unity. Our global object will be $f^s\varphi$. Consider what it looks like on V_{x_i}. We ignore h since it vanishes on all of X. Recall that $p_{x_i}q_{x_j} = p_{x_j}q_{x_i}$ implies $q_{x_j}(p_{x_i}/q_{x_i}) = p_{x_j}$.

$$\begin{aligned} f^s\varphi &= (\sum_i u_i q_{x_i}) \frac{p_{x_i}}{q_{x_i}} \\ &= u_i p_{x_i} + \sum_{j \neq i} u_j (q_{x_j} \frac{p_{x_i}}{q_{x_i}}) \\ &= u_i p_{x_i} + \sum_{j \neq i} u_j p_{x_j}. \end{aligned}$$

Therefore $\varphi \in k[X][1/f]$. □

Definition 1. Let $X \subseteq k^n$ be an irreducible algebraic subset. Then

$$k(X) = \varinjlim_{\varnothing \neq U \subseteq X} \Gamma(U, \mathcal{O}_X).$$

Remark 2.

- On the LHS is the set of rational functions on X. On the RHS is the set of equivalence classes $[(U, \varphi)]$ such that $(U_1, \varphi_1) \sim (U_2, \varphi_2)$ if there is $V \subseteq U_1 \cap U_2$ such that $\varphi_1 = \varphi_2$ on V.
- We may rewrite the direct limit in terms of distinguished open subsets, as they form a basis.

$$\begin{aligned} k(X) &= \varinjlim_{\varnothing \neq X_f \subseteq X} \Gamma(X_f, \mathcal{O}_X) \\ &= \varinjlim_{\varnothing \neq X_f \subseteq X} k[X][1/f] \\ &= \mathrm{Frac}(k[X]). \end{aligned}$$

- The theorem tells us that we may identify each ring of sections of regular functions with a subring of the rational functions on X.

$$\Gamma(U, \mathcal{O}_X) \subseteq k(X), U \neq \varnothing.$$

Definition 2. Let $x \in X$. The stalk of $\mathcal{O}_X$ at x is

$$\begin{aligned} \mathcal{O}_{X,x} &:= \varinjlim_{x \in U} \Gamma(U, \mathcal{O}_X) \\ &= \{\varphi \in k(X) \mid \varphi = [(U, \varphi)], x \in U\}. \end{aligned}$$

Remark 3.

- Observe that $\Gamma(U, \mathcal{O}_X) = \cap_{x \in U} \mathcal{O}_{X,x}$.
- The corresponding algebraic statement is $R = \cap_{x \in X} R_{\mathfrak{m}_x}$, where $\mathfrak{m}_x$ are the maximal ideals corresponding to the points x.