MATH 552 Algebraic Geometry I

1 09.19

Definition: An affine algebraic variety is a pair $(X, \mathcal{O}_X)$ where X is a topological space $\mathcal{O}_X$ is the sheaf of the k-valued functions on X, so that $(X, \mathcal{O}_X)$ is isomorphic to the affine algebraic set with its sheaf of regular function.

1.1 Category of sheaves

Given X a topological space, it makes a category $\mathbf{Top}(X)$, s.t. the objects are open sets and morphisms are inclusions.

Definition: A presheaf on X is a contravariant functor F, from $\mathbf{Top}(X)$ to some other category.

$$\begin{array}{ccc} U \subset X & \xrightarrow{F} & F(U) = \Gamma(U, \mathcal{O}_X) \\ \big\downarrow \supset & & \big\downarrow \text{group homomorphism} \\ V \subset X & \longrightarrow & F(V) \end{array}$$

Definition: A presheaf F on X is a sheaf, if:

$U = \cup_i U_i$,

- $s, t \in F(U)$ and $s|_{U_i} = t|_{U_i} \in F(U_i)$, then $s = t$.

- If $s_i \in F(U_i)$, $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for any i, j, then there exists $s \in F(U)$, s.t. $s|_{U_i} = s_i$.

Definition (Pushforward): $\pi : Y \to X$ of topological spaces,
F (pre) sheaf on $Y \to \pi_* F$ (pre) sheaf on X, for $U \subset X$, $(\pi_* F)(U) := F(\pi^{-1}(U))$.

1.2 Morphisms of affine algebraic sets

For $Y \subset k^m$, $X \subset k^n$,

$$\begin{array}{ccc} Y & \xrightarrow{F} & X \\ \downarrow & & \downarrow \\ (X, \mathcal{O}_X) & \xrightarrow{\sim} & (Y, \mathcal{O}_Y) \end{array}$$

if there exists $\pi : X \to Y$ is a homeomorphism s.t. the pullback of (pre) sheaves is iso.

Exercise: Show $\mathbb{C}^2 \setminus \{0\}$ with its sheaf of regular functions is not an affine variety.

Definition A prevariety is $(X, \mathcal{O}_X)$ so that:
- X is connected.
- $X = \cup_{i=1}^m U_i$ is an open cover, so that $(U_i, \mathcal{O}_{X|_{U_i}})$ is an affine variety.

Example: Glue affine together along affines.

Lemma: If $(X, \mathcal{O}_X)$ is a prevariety, then X is irreducible, Noetherian, can be broken into finitely many irreducible components, quasi-compact

Exercise: $X \in k^n$, $0 \neq f \in k[x_1, ..., x_n]$, check X_f is an affine algebraic variety.