Math 552, Algebraic Geometry
September 24, 2018 Lecture Recap

Remark. A topological space X is Hausdorff if and only if its diagonal $\Delta(X) \subseteq X \times X$ is a closed subset. However the topology on products of (pre)varieties is not the product topology.

Let X and Y be prevarieties over a field k. Then the product $X \times Y$ is a prevariety over k, and there exist two possible constructions.

1. (Hartshorne, II.3.3) The fibered product of arbitrary schemes exists: glue the products of affines.

2. Construct $X \times Y$ as a topological space (done by using the natural open cover of products of affines), and then put a sheaf $\mathscr{O}_{X\times Y}$ of k-valued functions on it. We define

$$\begin{aligned}\Gamma(U, \mathscr{O}_{X\times Y}) &= \{\varphi \in k(X) \otimes_k k(Y) : \text{for all } (x,y) \in U, \varphi \in \mathscr{O}_{X,x} \otimes_k \mathscr{O}_{Y,y}\} \\ &= \bigcap_{(x,y)\in U} \mathscr{O}_{X,x} \otimes_k \mathscr{O}_{Y,y}.\end{aligned}$$

We opt for the second construction (at least for now).

Subspaces. For an open subset $U \subseteq X$ of a prevariety X, U inherits the structure of a prevariety from the subspace topology, where $\Gamma(V, \mathscr{O}_U) := \Gamma(V, \mathscr{O}_X)$ for any open subset $V \subseteq U$.

For a closed subset $Z \subseteq X$ with the subspace topology, we may give Z the structure of a prevariety as well, by defining

$$\Gamma(V, \mathscr{O}_Z) := \{\varphi : V \to k : \forall x \in U \text{ there is open } V_x \subseteq X, V_x \ni x, \text{ and } \widetilde{\varphi} \in \Gamma(V_x, \mathscr{O}_X) \text{ s.t. } \widetilde{\varphi}|_{V_x\cap U} = \varphi|_{V_x\cap U}\}.$$

A subspace of a topological space is *locally closed* if it satisfies the following equivalent conditions: it is open as a subspace of its closure, or if it is the intersection of an open subspace and a closed subspace. The above implies that locally closed subspaces of prevarieties also have the structure of prevarieties.

Exercise. Let $\pi : Y \to X$ be a map of prevarieties. Then $\overline{\pi(Y)} \subseteq X$ is irreducible and closed. Check that $\pi(Y)$ is locally closed as follows: choose $U \subseteq X$ open affine, and check that $\overline{\pi(Y)} \cap U$ is $V(I)$, where $I = \ker(\Gamma(U, \mathscr{O}_X) \to \Gamma(\pi^{-1}(U), \mathscr{O}_Y)$.

Proposition. The following are equivalent for a prevariety X:

1. $\Delta(X) \subseteq X \times X$ is closed,

2. For any prevariety Y and maps $f, g : Y \to X$, the set $\{y \in Y : f(x) = g(x)\}$ is closed.

Definition. A *variety* over the field k is a prevariety satisfying the above equivalent conditions.

Example. Affine varieties (in the old sense) are varieties.