# MATH 552 ALGEBRAIC GEOMETRY I

SEPTEMBER 28 RECAP

Lemma Let $X$ be a variety, and let $U, V \subseteq X$ be affine open subsets. Then $U \cap V$ is also affine.

Example (Graph of a morphism) Let $\pi : Y \to X$ be a map of varieties. Then one can construct $\Gamma_\pi = \{(y, x) | \pi(y) = x\} \subseteq Y \times X$, the graph of $\pi$. $\Gamma_\pi$ is closed in $Y \times X$.

Nonexample 1 Let $X$ be the gluing together of two copies of $k^2$ along $k^2 \setminus \{0\}$, yielding a "plane with the origin doubled." Then $X = U_1 \cup U_2$ with each $U_i \cong k^2$ and $U_1 \cap U_2 = k^2 \setminus \{0\}$, so $X$ is not a variety, but is still a prevariety.

Nonexample 2 Similarly, construct $X$ to be a "line with the origin doubled" where $X = U_1 \cup U_2$ with each $U_i \cong k$ and $U_1 \cap U_2 = k \setminus \{0\}$. One can check that this is not a variety by considering the maps from $k$ onto each of the $U_i$, which don't agree on $k \setminus \{0\}$.

Rational Normal Curves (of degree $d$ in $\mathbb{P}^d$) Two ways to look at this:

**(In coordinates)** Let $X$ be the image of the map from $\mathbb{P}^1$ to $\mathbb{P}^d$ given by $[s : t] \mapsto [s^d : s^{d-1}t : \cdots : t^d]$. This map is an isomorphism onto its image, and on affines looks like a map from $k$ to $k^d$ given by $t \mapsto (t, t^2, \ldots, t^d)$. One might be able to check that $X$ is closed by realizing $X = \mathbb{V}\left(2 \times 2 \text{ minors of } \begin{bmatrix} x_0 & \ldots & x_{d-1} \\ x_1 & \ldots & x_d \end{bmatrix}\right)$.

Exercise Check or fix the above realization of $X$ and show that the given map is really an isomorphism onto its image.

Exercise Let $F(x_0, \ldots, x_n)$ denote the determinant of the Vandermonde matix $\begin{bmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \ldots & x_n^n \end{bmatrix}$.

(a) Check that $F$ is homogeneous. What is its degree? Show that

$$F = \prod_{0 \leq i < j \leq n} (x_j - x_i)$$

and conclude that $\mathbb{V}(F) \subseteq \mathbb{P}^n$ is a union of hyperplanes.

(b) Check that any $d + 1$ distinct points on the rational normal curve in $\mathbb{P}^d$ are in general position.

**(Forget coordinates)** Let $V$ be a vector space of dimension 2 and consider the map from $\mathbb{P}(V)$ to $\mathbb{P}(\mathrm{Sym}^d(V))$ such that if you pick coordinates on $\mathbb{P}(V)$, you get $[s : t] \mapsto$ [basis of vector space of homogeneous polynomials of degree d in s and t].

Proposition Through any $d + 3$ points in general position in $\mathbb{P}^d$, there exists a unique rational normal curve.