Math 552, Algebraic Geometry
September 7, 2018 Lecture Recap

Remark. We previously discussed the so-called standard affine cover of projective space $\mathbb{P}^n_k = U_0 \cup \cdots \cup U_n$, where $U_i = (x_i \neq 0)$. This gives rise to a decomposition of the projective space into affine spaces: observe that, as sets,

$$\mathbb{P}^n_k \setminus U_0 = \{[a_0 : \cdots : a_n] \mid a_i = 0\} \simeq \mathbb{P}^n_k$$

so that repeating this description, we have

$$\mathbb{P}^n = k^n \amalg k^{n-1} \amalg \cdots \amalg k^1 \amalg k^0.$$

Definition. For an affine algebraic set $X \subseteq k^{\oplus n}$, the identification $k^{\oplus n} \simeq U_0 \subseteq \mathbb{P}^n_k$ gives rise to the *projective closure* $\overline{X} \subseteq \mathbb{P}^n_k$ of X in $\mathbb{P}^n_k$.

Definition. For a polynomial $0 \neq f \in k[x_1, ..., x_n]$ we define the *homogenization of f* (in the variable x_0) by $\tilde{f} = x_0^{\deg(f)} f(x_1/x_0, ..., x_n/x_0) \in k[x_0, ..., x_n]$. For $0 \neq G \in k[x_0, ..., x_n]$ we define the *dehomogenization of G* (in the variable x_0) by $g = G(1, x_1, ..., x_n) \in k[x_1, ..., x_n]$.

Remark. Note that $\deg(g) \leq \deg(G)$, and that $\deg(\tilde{f}) = \deg(f)$.

We now describe $\overline{X}$:
Case 1. X is a hypersurface, i.e., $X = V(f) \subset k^{\oplus n}$. Then $\overline{X} = V(\tilde{f})$.
Case 2. The general case: $X = V(I)$. Then $\overline{X} = V(\{\tilde{f} \in k[x_0, ..., x_n] \mid f \in I(X)\})$.

Remark/example (the twisted cubic). If $X = V(f_1, ..., f_r)$, it does not suffice to take $\overline{X} = (\tilde{f}_1, ..., \tilde{f}_r)$. Consider $X = \{(t, t^2, t^3) \in k^{\oplus 3} : t \in k\} = V(x - y^2, z - xy)$. Then $\overline{X} \neq (wy - x^2, wz - xy)$, as $[0 : 0 : 1 : 1] \in \overline{X}$ (and many others) does not satisfy the equation $xz - y^2 \in I(X)$.

Exercise. Prove the statement in Case 2.

Definition. For affine algebraic sets $Y \subseteq k^{\oplus m}$, $X \subseteq k^{\oplus n}$, a map $\varphi : Y \to X$ is *regular* (or *polynomial*) provided that it is the restriction of some $\widetilde{\varphi} : k^{\oplus m} \to k^{\oplus n}$ whose coordinate functions are polynomial. Equivalently, the coordinate functions $y_i \in k[y_1, ..., y_n]$ pull back to polynomial functions in $k[x_1, ..., x_m]$, i.e., the precomposition $y_i \circ \varphi$ lies in $k[x_1, ..., x_m]$. Or, equivalently, pull back takes $k[X] \to k[Y]$.

Observation/Proposition. There is an equivalence of categories

$$\left\{ \begin{array}{c} \text{affine algebraic sets} \\ \text{with regular maps} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{reduced, finitely generated } k\text{-algebras} \\ \text{with } k\text{-algebra maps} \end{array} \right\}$$

taking

$$X \longmapsto k[X]$$

$$\left(\begin{array}{l} R \simeq k[\mathbf{x}]/I \\ V(I) = X \end{array} \right) \longleftarrow R$$

$$(\varphi : Y \to X) \longmapsto \text{pullback: } k[X] \to k[Y]$$

$$(V(I) \leftarrow V(J)) \longleftarrow (R \simeq k[\mathbf{x}]/I) \to (S \simeq k[\mathbf{x}]/J)$$

Exercise. Complete the following.

(a) Describe the image of the regular map $(x, y) \mapsto (x, xy)$: is it open? closed? dense?

(b) Assume that $\operatorname{char}(k) \neq 2$. Let V_c denote the intersection of the affine algebraic sets $x^2 + y^2 - 1 = 0$ and $x - c = 0$. Find $I(V_c)$. Two choices of c are special; which ones? These will be called "ramification points." Consider the map $k^{\oplus 2} \to k$ given by $(x, y) \mapsto x$. "Draw" the map, and interpret the special c's geometrically.

(c) Repeat (b) in the setting $\operatorname{char}(k) = 2$.

(d) Give the induced maps on coordinate rings.