①

Surfaces w/ infinite-dimensional CH_0

Recall $CH_0(X) = \bigoplus_{p \in X} \mathbb{Z}\, p \Big/ \text{rat equiv}$

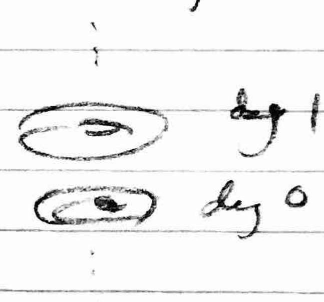
where rat equiv gen by relations of form $\sum p_i = \sum q_i$ whenever $\exists$ $T \subseteq X \times \mathbb{P}^1$ w/ $\pi^{-1}(\{0\}) = \bigcup p_i$, $\pi^{-1}(\{\infty\}) = \bigcup q_i$

$$\begin{array}{c} T \subseteq X \times \mathbb{P}^1 \\ \big\downarrow \text{flat} \\ \mathbb{P}^1 \end{array}$$

Q: How big is CH_0? A: Often very big

Ex: a) $X = \mathbb{P}^n \implies CH_0 X = \mathbb{Z}$
(any 2 pts joined by line)

b) X = elliptic curve
$CH_0 X = \mathrm{Pic}(X)$



" tower of finite dim vars "

c) X genus g curve

$$\mathrm{Pic}^0 X \longrightarrow CH_0(X) \longrightarrow H_0(X) = \mathbb{Z}$$

$\mathrm{Pic}^0 X = CH_0(X)_{hom}$ (same picture)

"**Def**" $\mathrm{Sym}^k X \twoheadrightarrow CH_0(X)_{\deg k}$

$\Longleftrightarrow$

$CH_0(X)$ is finite dim

Def Say X has fin dim $CH_0(X)$ if $\exists$

(2) c st $\forall$ $k, A \in \mathrm{Sym}^k X$ $\exists$

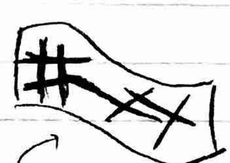
$Z \subseteq \{B \sim A\}$ subvar w/ $\mathrm{codim}(Z \subseteq \mathrm{Sym}^k X) \leq c$.

Pf: "$\Uparrow$" $\{B \sim A\}$

$$\begin{array}{ccccc} & \{B \sim A\} & & & \\ & \cup\!\mid & & & \\ A = \Sigma p_i - \Sigma q_i & Z & \cap & \{A' + q_1\} & \leftarrow \text{must } \cap \text{ for } k \gg 0. \\ & \text{codim} \leq c & & \dim = (k-1)\dim X & \end{array}$$

Q: Is $CH_0(X)$ always finite dimensional?
A: Not always!

Thm$/\mathbb{C}$ (Mumford): X smooth projective surface w/ $H^0(\Omega^2_X) \neq 0$, Then $CH_0(X)$ is not finite dimensional.

Ex:  X K3, $\Omega^2_X = \mathcal{O}_X$

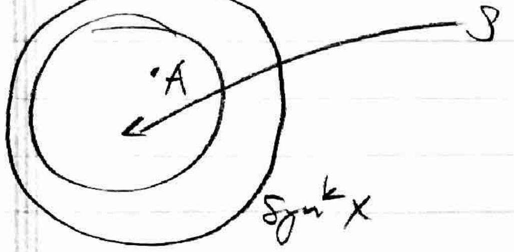
$H^0(\Omega^2_X) \neq 0$

Has countable many rat curves, all As on them equivalent

Idea: use $0 \neq \omega \in H^0(\Omega^2_X)$ to construct a differential 2-form on $\mathrm{Sym}^k X$ that must pull back to any $S \subseteq \mathrm{Sym}^k X$ of equivalent 0-cycles.

"Pf:" Let $S \longrightarrow \mathrm{Sym}^k(X)$ be a map from S smooth to a space of equivalent 0-cycles. Let $0 \neq \omega \in H^0(\Omega^2_X)$

(3)



$$\begin{tikzcd} N \arrow[r, "\text{normalize}"] \arrow[dr, bend right, "\varphi"'] & \tilde{S}_{red} \arrow[r, "\tilde f"] \arrow[d, "p"] & X^k \arrow[d] \\ & S \arrow[r, "f"] & Sym^k X \end{tikzcd}$$

$\underbrace{\omega^{[k]} = \sum p_i^* \omega}_{\text{descends}}$

Lemma: $\exists!$ $\eta_f \in H^0(S, \Omega_S^2)$ with $\tilde f^*(\omega^{[n]}) - p^*(\eta_f)$ torsion

Lemma: η_f is functorial in f, ie if $f: S \to Sym^n X$, $g: S' \to S$, $\eta_{g \circ f} = g^* \eta_f$.

Lemma: η_f is additive in f, ie if $f: S \to Sym^k X$, $g: S \to Sym^m X$, $\eta_{f+g} = \eta_f + \eta_g$.

Thm Let $f: S \to Sym^k(X)$ be a map w/ all zero cycles in image rationally equiv. Then $\eta_f = 0$.

Pf: $A \in$ im f.

Lemma: $\exists$ T smooth with $e: T \to S$ dominant, $g: T \to Sym^m(X)$, $h: T \times \mathbb{P}^1 \to Sym^{k+m}(X)$ with $h(t, 0) = f(e(t)) + g(t)$, $h(t, \infty) = A + g(t)$

(4)

$$\eta_h \big|_{T \times \{0\}} = \eta_g + e^* \eta_f$$

$$\eta_h \big|_{T \times \{\infty\}} = \eta_g$$

$$\Omega^2_{T \times \mathbb{P}^1} = p_1^* \Omega^2_T + p_2^* \Omega^1_{\mathbb{P}^1} \otimes p_1^* \Omega^1_T$$

$$\eta_h = p_1^* \omega \quad \text{for } \omega \in H^0(T, \Omega^2_T)$$

$\Rightarrow$ η_h constant as $\mathbb{P}^1$-parameter varies

$$\Rightarrow \quad \eta_g = \eta_g + e^* \eta_f \Rightarrow e^* \eta_f = 0 \xrightarrow[\text{edom}]{} \eta_f = 0. \quad \square$$

Sketch of Mumford:

$\sum x_i = A \in \mathrm{Sym}^k X$ smooth

ω nondegen on x_i's

$\Rightarrow$ $\omega^{[k]} \rightsquigarrow \omega'$ nondegen in nbhd of A

$$\begin{tikzcd}
 & X^k \arrow[d] & \omega^{(k)} \arrow[d, squiggly] \\
S \arrow[r] & \mathrm{Sym}^k X & \omega' \\
\eta_f \arrow[urr, bend right] & &
\end{tikzcd}$$

$$0 = \eta_f = f^* \omega'$$

$$\omega' \big|_{T_A f(S)} = 0 \Rightarrow \dim T_A f(S) \leq k$$

$$\Rightarrow \dim f(S) \leq k.$$