(1)

Unirational K3 surfaces

Def X K3 surface / k

$\Longleftrightarrow$

X sm proj / $k = \bar{k}$, $\omega_X \cong \mathcal{O}_X$, $H^1(X, \mathcal{O}_X) = 0$

E.g. Fermat quartic $X_4 = \mathbb{V}(x^4 + y^4 + z^4 + t^4) \subseteq \mathbb{P}^3_k$

Def: A variety X of dimension n is unirational if $\exists$ $\mathbb{P}^n \dashrightarrow X$ dominant, generically finite; $k(X)$ admits a finite ext L such that L is a purely transc ext of k of trdeg n.

E.g. (Tate '65) X_4 has max Picard rank $\rho(X_4) = 22$ over any k w/ char $k \equiv 3$ (4)

(Artin - Shioda)

$$\begin{tikzcd}[column sep=large, row sep=large]
\text{Artin supersingular (def complicated)} \arrow[rr, bend left, "\text{+ elliptic, or Tate's conjecture true}"] \arrow[dr, dashed, "\text{Artin (conj) Rudakov-Shafarevich, Shioda}"] & & \text{Shioda supersing (def } \rho(X) = 22) \arrow[ll, "\text{Artin}"'] \arrow[dl, dashed, "\text{(conj)}"'] \\
& \text{unirationality} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

($p \geq 5$; all notions of supersingular known to be equiv)

[Maulik, Liedtke, Lieblich

(2) (Shioda)

Prop $X_n = V(x^n + y^n + z^n + t^n) \subseteq \mathbb{P}^3_k$ is unirational $\forall\ n \not\equiv 0 \pmod{p}$ st $\exists\ q = p^\nu$ w/ $q \equiv -1 \pmod{n}$

E.g. In particular, X_4 unirational over $\overline{\mathbb{F}}_p$ for $p \equiv 3\,(4)$.

Pf: Pf for $n=4$, $p=3$.

$$x^4 - y^4 = z^4 - t^4$$

put
$$\left.\begin{array}{ll} y_1 = x+y & y_2 = x-y \\ y_3 = z+t & y_4 = z-t \end{array}\right\} \Rightarrow$$

$$y_1 y_2 (y_1^2 + y_2^2) = y_3 y_4 (y_3^2 + y_4^2)$$

Set $y_4 = 1$, $y_2 = y_1 u$, $y_3 = uv$

$$K(X) \cong k(y_1, u, v) \big/ (f)$$

$$f:\quad y_1^4(1+u^2) = v(u^2v^2+1)$$

$$y_5 = y_1^{1/3} \Rightarrow K' = k(y_5, u, v)$$

(3)

purely insep / $K(X)$

$$u^2(y_4^4 - v)^3 = v - y_5^{12}$$

$$s = u(y_5^4 - v)$$

$$\Rightarrow v = \frac{y_5^4(s^2 + y_5^8)}{s^2+1}$$

$$K' \cong k(y_5, s) \cong K(\mathbb{P}^2_k)$$

$\Rightarrow X$ unirational. ✓

<u>Rmk</u> X_4 not unirational for char $k = p \equiv 1\ (4)$.

N.B. In general, $\rho(X) \leq b_2(X)$

$$b_2(X) - \rho(X) = \lambda(X) \qquad H^2_{et}(X, \mathbb{Q}_\ell)$$

Lefschetz # $\qquad \ell \neq p$ prime

<u>Def</u> (Shioda supersingular) An alg surface Y is supersingular if $\lambda(Y) = 0$

(4)

<u>Prop 2</u> Unirational $\Longrightarrow$ (Shioda) super sing

<u>Lemma</u> $f : Y \dashrightarrow Z$ gen surjective, Y, Z sm surfaces

$\Rightarrow \lambda(Y) \geq \lambda(Z)$.

<u>Pf of Lemma</u>: Can replace f by a morphism $f : Y \twoheadrightarrow Z$.

$$S_Y \subseteq H^2(Y) = \begin{cases} H^2(Y, \mathbb{Q}) & \text{if char } 0 \\ H^2_{et}(Y, \overline{\mathbb{Q}}_\ell) & \text{if char } p \\ & \ell \neq p \end{cases}$$

subspace spanned by alg cycles

$$f^* : H^2(Z) \longrightarrow H^2(Y)$$

$$f_* : H^2(Y) \longrightarrow H^2(Z)$$

$$f_* f^*(x) = (\deg f) \cdot x$$

$$f^*(S_Z) \subseteq S_Y \quad , \quad f_*(S_Y) \subseteq S_Z$$

$\Rightarrow f^*$ induces an injection

$$H^2(Z)/S_Z \hookrightarrow H^2(Y)/S_Y$$

$$\lambda(Z) \leq \lambda(Y).$$

5

Pf of prop 2: $\lambda(\mathbb{P}^2) = 0$ ✓

1. Over arbitrary field:

$$\text{unirational} \longrightarrow \begin{cases} \text{super sing} \\ \text{Alb}(X) = 0 \end{cases}$$

2. char 0: super-sing $\Longleftrightarrow$ $P_g = 0$

Q: char $p > 0$, X super sing, $P_g = q = 0$

$\overset{?}{\Rightarrow}$ unirational

A: No. $X_5 \subseteq \mathbb{P}^3$, $X = X_5 / (\mathbb{Z}/5\mathbb{Z})$

$\mathbb{Z}/5\mathbb{Z} = \langle \zeta \rangle$

For $p \equiv 1 \pmod 5$, $\zeta : (x_i) \longmapsto (\zeta_5^i x_i)$

$X_{(p)} = X$ in char p

For $p \equiv 1\ (5)$ $X_{(p)}$ $P_g = q = 0$ s.s

$p \not\equiv 1\ (5)$ $X_{(p)}$ unirational

Statement of Tate's Conjecture

k = field, X smooth geom irred proj var $/k$

$\dim X = d$, $\overline{X} = X \times_k \overline{k}$

$G = \mathrm{Gal}(\overline{k}/k)$

$Z^r(\overline{X})$ = free abel gp of codim r cycles on $\overline{X}$

$\cup\!|$

$Z^r(X)$ " " on X

$$c^r : Z^r(\overline{X}) \longrightarrow H^{2r}_{et}(\overline{X}, \mathbb{Q}_\ell(r))$$

A^r = image of c^r $\qquad (\mathbb{Z}_\ell(1) = \varprojlim_n \mu_{\ell^n})$

$$\begin{tikzcd} A^r(\overline{X}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \arrow[r, "c^r \otimes 1"] & H^{2r}_{et}(\overline{X}, \mathbb{Q}_\ell(r)) \arrow[d] \\ A^r(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \arrow[u, hook] \arrow[r, "\varphi_r"] & H^{2r}_{et}(\overline{X}, \mathbb{Q}_\ell(r))^G \end{tikzcd}$$

E.g. $(r=1)$ $\quad 0 \to \mu_{\ell^n} \to \mathbb{G}_m \xrightarrow{\ell^n} \mathbb{G}_m \to 0$

$$c^1 : Z^1(\overline{X}) \to \mathrm{Pic}(\overline{X}) \to \mathrm{Pic}(\overline{X}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \to H^2(\overline{X}, \mathbb{Q}_\ell(1))$$

7

<u>Conj</u> (Tate) - $T^r(X)$: φ_r is bijective

- $E^r(X)$: $A^r(\overline{X})$ indep of l

Tate's conjecture $\Rightarrow$ $\left(\begin{array}{c} X \quad K3, \quad \rho(X) = 22 \\ \Downarrow \\ \text{unirational} \end{array} \right)$