Introduction (John Lesieutre)

X sm proj var over $k = \bar{k}$.

$Z(X)$ = group of cycles on X, generated by subvarieties of X

this is graded by dimension (or codim)

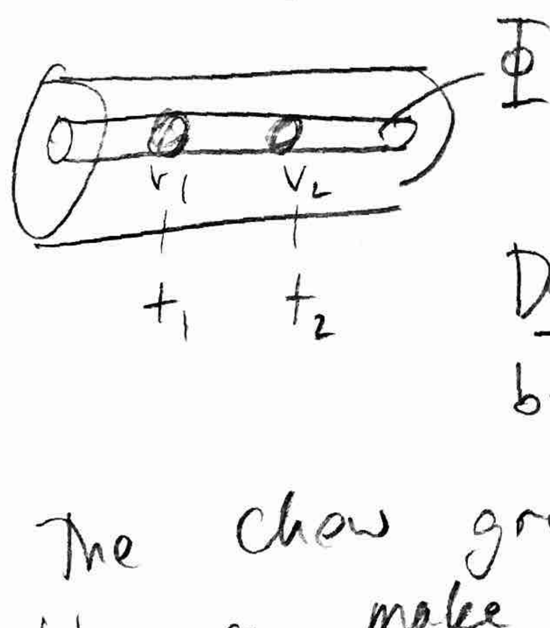
$Z(X) = \bigoplus Z_k(X)$, Z_k generated by k-dim'l stuff

Rational Equivalence

$V_1, V_2 \subseteq X$ are rationally equivalent if there's a rationally parameterized family connecting them $\Phi \subseteq X \times \mathbb{P}^1$ s.t.

$\Phi \cap (X \times \{t_0\}) = V_1$

$\Phi \cap (X \times \{t_1\}) = V_2$



Def $Rat(X) \subseteq Z(X)$ generated by $(\Phi \cap \{X \times \{t_0\}\}) - (\Phi \cap \{X \times \{t_1\}\})$.

The chow group is $A(X) = Z(X) / Rat(X)$.
We can make $A(X)$ a ring in such a way that if $Y, Z \subseteq X$ intersect generically transverse, then $[Y] \cdot [Z] = [Y \cap Z]$. Graded by codim.

Ex $X = \mathbb{P}^n$. Then $A(X) = \mathbb{Z}[H] / \langle H^{n+1} \rangle$

$H \in A^1(X)$ class of hyperplane.

codim 1

If V has deg d codim k,

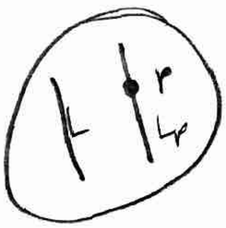
$$[V] = d H^k.$$

Moving Lemma Given $\alpha, \beta \in A(X)$, can find A_i, B_j so $\alpha = \sum m_i A_i$, $\beta = \sum n_j B_j$ and A_i, B_j generically transverse.

Ex $X = Bl_p \mathbb{P}^2$. $A_1(X)$ spanned by E and L.

exceptional

line not thru p



$L \sim L_p + E$

$$\begin{aligned} E \cdot E &= (L - L_p) \cdot E \\ &= 0 - 1[pt] \\ &= -1\,[pt] \end{aligned}$$

Functoriality for $A(X)$ $f : Y \longrightarrow X$ a morphism induces $f_* : A(Y) \longrightarrow A(X)$. Given $V \subseteq Y$, set

$$f_*([V]) = \begin{cases} 0 & \text{if } \dim f(V) < \dim V \\ (\deg f|_V)\,[f(V)] & \text{if } \dim f(V) = \dim V \end{cases}$$

<u>Ex</u> $f: Bl_p\mathbb{P}^2 \longrightarrow \mathbb{P}^1$ so $f_*([L_p])$

$f_*([L]) = [\mathbb{P}^1]$ $= f_*([L]) - f_*([E])$

$= 0$

$f_*([E]) = [\mathbb{P}^1]$

(as it should! L_p is a fiber of f!)

<u>Ex</u> $\mathbb{G}(1,3) = \{\text{lines in } \mathbb{P}^3\}$

$A(\mathbb{G}(1,3)) = \mathbb{Z}[\sigma_1, \sigma_2] / \langle \sigma_1^3 - 2\sigma_1\sigma_2, \; \sigma_1^2\sigma_2 - \sigma_2^2 \rangle$

$\sigma_1 = \begin{bmatrix} \text{lines meeting a} \\ \text{fixed line} \end{bmatrix}$ codim 1

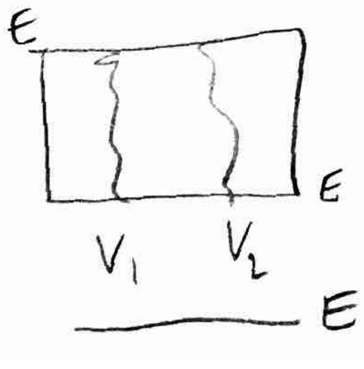
$\sigma_2 = \begin{bmatrix} \text{lines thru a} \\ \text{given point} \end{bmatrix}$ codim 2

(so $A(X)$ not always generated by divisors)

<u>Other equivalence relations:</u>

1) Algebraic equivalence

($V_1 \sim V_2$ if connected by a family over any curve C)

e.g.  $E \times E$ (V_1, V_2 two fibers)

E elliptic curve

2) Numerical equivalence

$V_1 \sim V_2$ if $\deg(V_1 \cdot W) = \deg(V_2 \cdot W)$ for any W of complimentary dimension.

<u>Fact</u>: $N(X)$ is finitely generated!

Equivalence relations for divisors

1) $Z^1(X)/\mathrm{Rat}(X)$ is divisors / linear equivalence = Picard Scheme

2) $Z^1(X)/\mathrm{Alg}$ is $NS(X)$, fin gen group of components of $Z^1(X)/\mathrm{Rat}(X)$

3) $Z^1(X)/\mathrm{Num}$ is $NS(X)/\text{torsion}$

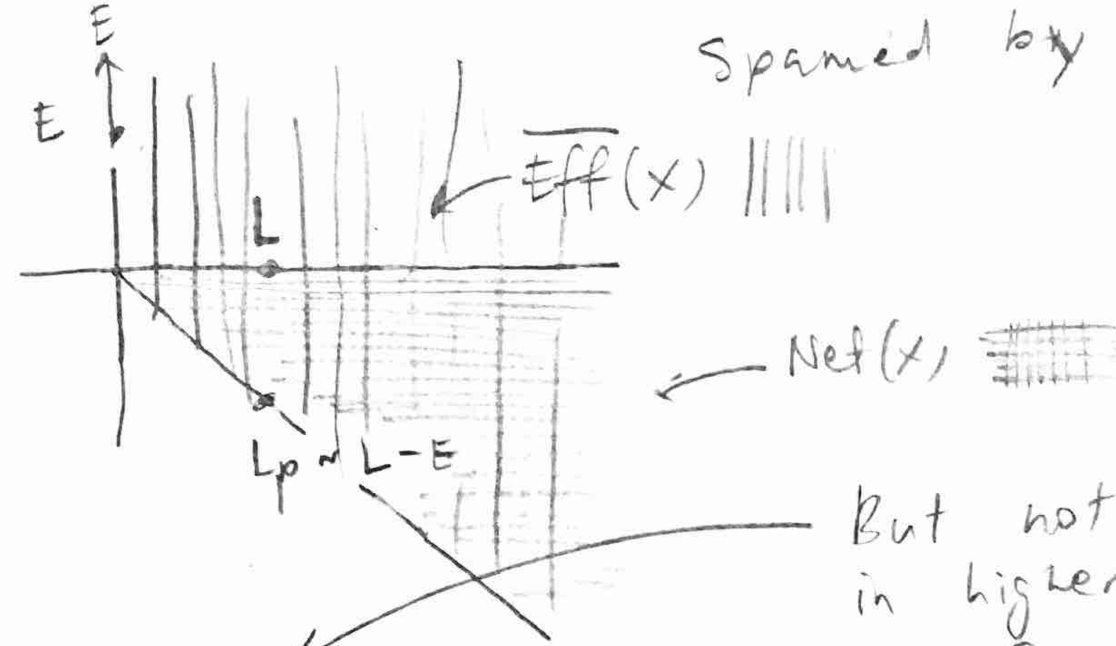
Cones of Effective and Nef Cycles

Inside $N_k(X) \otimes \mathbb{R}$, look at

1) $\overline{\mathrm{Eff}}_k(X)$ = closure of cone generated by effective cycles

2) $\mathrm{Nef}_k(X)$ = cone of nef cycles: V is nef if $V \cdot W \geq 0$ for any effective W of complimentary dimension.

Ex $X = Bl_p \mathbb{P}^2$ $N^1(X)$ is 2-dim
Spanned by L, E



But not nec / tricky in higher codim ...

For divisors: • the interior of $Nef^1(X)$ is $Amp(X)$, cone generated by ample divisors. (Kleiman)

($\Rightarrow$ any nef divisor is pseff, since any ample is effective)

($\Rightarrow$ Nef(X) has full dimension)

• If D is basepoint free, it's nef.

(e.g. contrast w/
$X = Bl_p \mathbb{P}^4$, $S \subseteq X$ a $\mathbb{P}^2$ in exceptional div.
Given any $x \in X$, can find $S' \equiv S$ not thru x. But $S \cdot S = -1$)

<u>Ex</u> E general elliptic curve, $X = E \times E$.

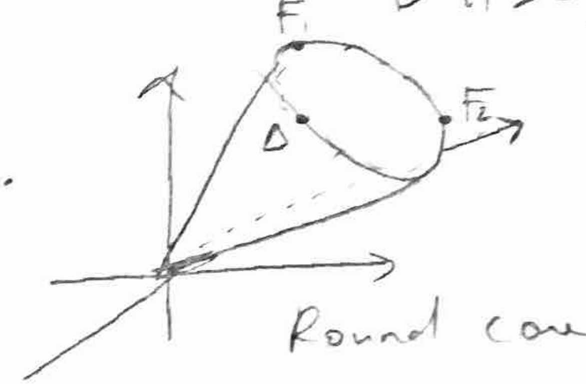
If D is effective, $D^2 \geq 0$. If $D^2 > 0$ then (some mult of) D is effective (Riemann-Roch).

$(+\, D \cdot H > 0$ w/ H ample$)$

$\therefore\ \overline{\mathrm{Eff}}(X) = \{ D \mid D^2 \geq 0,\ D \cdot H \geq 0 \}$

$N^1(X)$ is 3-dimensional, spanned by F_1, F_2, Δ.

Which boundary classes are effective?



$SL_2(\mathbb{Z})$ acts on X $\rightsquigarrow$ there are countably many effective things on boundary.

But also uncountably many non-effective ones too! (these are pseudoeffective)

<u>Another Ex</u> $X = Bl_{16}\mathbb{P}^2$, $D = 4H - E_1 - \cdots - E_{16}$

$H^0(X, D) = 0$, too many pts not enough quartics.

Is $H^0(X, nD) \neq 0$ for $n \gg 0$?

$H^0(X, nD) \Leftrightarrow$ degree $4n$ polynomials w/ mult n at 16 pts

$$\text{exp dim} = \frac{(4n+2)(4n+1)}{2} - \sum_1^{16} \frac{n(n+1)}{2} = 1 - 2n < 0$$

Can show $H^0(X, nD) = 0 \ \forall \ n > 0$.
But $D + \varepsilon A$ has $(D + \varepsilon A)^2 > 0$,
hence a multiple has sections. So D
is pseudoeffective.