HOMOLOGICAL CONJECTURES AND LIM COHEN-MACAULAY SEQUENCES[1]

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Abstract. We discuss the new notion of a lim Cohen-Macaulay sequence of modules over a local ring, and also a somewhat weaker notion, as well as the theory of content for local cohomology modules. We relate both to the problem of proving the direct summand conjecture and other homological conjectures without using almost ring theory and perfectoid space theory, and we also indicate some other open problems whose solution would yield a new proof of the direct summand conjecture.

This paper is dedicated to Winfried Bruns on the occasion of his 70th birthday, and celebrates his many contributions to commutative algebra.

1. Introduction

Throughout, all rings are commutative, associative, with identity, and homomorphisms are assumed to preserve the identity. By a *local ring* $(R, \mathfrak{m}, K)$ we mean a Noetherian ring R with a unique maximal ideal $\mathfrak{m}$ and residue class field K.

Since the time the talk on which this paper is based was given, the direct summand conjecture has been proved: see [2, 3, 10]. André's paper [3] includes a proof of the existence of big Cohen-Macaulay algebras, and contains the assertion that one can prove a weakly functorial version of the existence of big Cohen-Macaulay algebras for injective local maps of complete local domains, but does not give details. Bhatt's paper [10] gives a short proof of both the direct summand conjecture and a derived form of the conjecture introduced by de Jong. Bhatt's approach uses part of André's work, but is also substantially different: Bhatt utilizes a quantitative form of Scholze's Hebbarkeitssatz (a perfectoid version of the Riemann extension theorem) instead of André's generalizaton (the pefectoid Abhyankar lemma) of the almost purity theorem due to Faltings. We discuss some conjectures involving big Cohen-Macaulay algebras further in §14.

Despite all these exciting events, it would be desirable to have a proof of the direct summand conjecture that does not depend on almost ring theory and perfectoid space theory.

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We shall describe here nine conjectures each of which is stronger than the direct summand conjecture. There is a recap of these conjectures in §14. Despite the breakthroughs made in [2, 3, 10], none of these conjectures is known. Among these, we emphasize the conjecture that complete local domains have lim Cohen-Macaulay sequences, which are in introduced in [11] and defined in this manuscript in §5. In [11] it is shown that lim Cohen-Macaulay sequences exist in characteristic $p > 0$. It is also shown there that the existence of lim Cohen-Macaulay sequences in mixed characteristic implies the general case of the positivity of Serre intersection multiplicities, a question that has been open for over fifty years. The new developments using almost ring theory and perfectoid geometry as yet do not seem to help with Serre's multiplicity conjecture.

Although we emphasize here the theory of lim Cohen-Macaulay sequences, we also describe approaches that involve the theory of content of local cohomology modules and Q-sequences (see §10) developed in [48], [52], as well as a conjecture that involves tight closure closure theory in positive characteristic in §12.

For further background on the direct summand conjecture and related local homological conjectures we refer the reader to [4, 5, 8, 9, 19, 20, 21, 22, 23, 24, 25, 26, 27, 32, 33, 34, 35, 36, 37, 40, 41, 42, 43, 45, 47, 48, 54, 57, 58, 59, 63, 64, 65, 66, 67]. Basic references for almost ring theory are [28, 30] and [68] is a basic reference for the theory of perfectoid spaces. For treatments of other notions from commutative algebra not explained explicitly here, we refer the reader to [15], [56], and [69].

2. Big and small Cohen-Macaulay modules and algebras: a first glance

The existence of big Cohen-Macaulay modules (defined below) implies certain local homological conjectures, but not Serre's conjecture on multiplicities. The existence of small Cohen-Macaulay modules also implies Serre's conjecture. Hence, small Cohen-Macaulay modules are better.

But big Cohen-Macaulay modules are known to exist in equal characteristic, while not much is known about existence of small Cohen-Macaulay modules.

We discuss a new notion, introduced in a joint paper [11] with Bhargav Bhatt and Linquan Ma: lim Cohen-Macaulay sequences. The existence of these for complete local domains implies the existence of big Cohen-Macaulay modules and Serre's conjecture on multiplicities. And they do exist in positive characteristic!

We also discuss an alternate notion of lim Cohen-Macaulay sequences. The requirements are weaker, but using ideas related to the theory of *content* of local cohomology modules, developed in [48, 52], the existence of the lim Cohen-Macaulay sequences in this weakened sense is sufficient to deduce the direct summand conjecture.

By a *local ring* $(R, \mathfrak{m}, K)$ we mean a Noetherian ring R with a unique maximal ideal $\mathfrak{m}$ and residue class field K. Often, we assume that given rings are complete local domains, which suffices in all the applications we have in mind.

For simplicity we also freely assume that K is perfect, while this is not always needed, this case likewise suffices for applications.

If $(R, \mathfrak{m}, K)$ is local and M is finitely generated, $\nu(M) := \dim_K(K \otimes_R M)$, the least number of generators of M as an R-module. $\ell(H)$ is the length of the finite length module H. If R is a domain, $\mathrm{frac}(R) = \mathcal{F}$ and M is an R-module,

$\mathrm{rank}(M) = \dim_{\mathcal{F}}\mathcal{F}\otimes_R M$, the *torsion-free rank of* M. If R is also local, $\mathrm{rank}(M) \leq \nu(M)$.

If $f, g : \mathbb{N}_1 \to (0, \infty)$ where $\mathbb{N}_1 \subseteq \mathbb{N}$ contains all $n \gg 0$, $f(n) = O\big(g(n)\big)$ if $f(n)/g(n)$ is bounded and $f(n) = o\big(g(n)\big)$ if $f(n)/g(n) \to 0$ as $n \to \infty$.

A *big Cohen-Macaulay module* over a local ring $(R, \mathfrak{m}, K)$ is a (not necessarily finitely generated) module M such that $\mathfrak{m}M \neq M$ and every system of parameters for R is a regular sequence on M.

If M is finitely generated, then M is a big Cohen-Macaulay module for R iff $M \neq 0$ and one system of parameters is a regular sequence on M.

In this case, M is called a *small* Cohen-Macaulay module for R. Big Cohen-Macaulay modules (and algebras) exist in equal characteristic and and if $\dim(R) \leq 3$. This was shown for modules in [37], and for algebras in [42], and by other methods later in [45, 47]. The existence of big Cohen-Macaulay algebras in dimension 3 in mixed characteristic is shown in [43], based on the results of [33]. The latter results are improved in [34].

Small Cohen-Macaulay modules are only known to exist if $\dim(R) \leq 2$ (this is easy: if R is a complete domain of dimension at most 2, the integral closure is Cohen-Macaulay) or if R is $\mathbb{N}$-graded over a perfect field of characteristic p and has an isolated non-Cohen-Macaulay point at the origin: this was first observed by R. Hartshorne, and rediscovered independently by Peskine and Szpiro. A proof is given in [38]. While there are a few other special results (see, for example, [51]), there has not been much progress in 40 years!

In the early 1970s I conjectured:

Conjecture 2.1. *Every complete local domain has a small Cohen-Macaulay module.*

In the 2000s I conjectured:

Conjecture 2.2. *There are complete local domains that do not have a small Cohen-Macaulay module.*

One of these conjectures is bound to be correct.

The existence of big Cohen-Macaulay modules suffices to prove many of the local homological conjectures formulated by Maurice Auslander, Hy Bass, and, later by C. Peskine and L. Szpiro. See [4, 5, 57, 58]. Peskine and Szpiro proved a number of these conjectures in characterisc $p > 0$ and in certain cases in equal characteristic 0. I added the direct summand conjecture (regular local rings are direct summands of their module-finite extension rings) and some others [36, 41], and showed that big Cohen-Macaulay modules exist in equal characteristic Ho3, thus settling a number of these conjectures in the general equicharacteristic case. Paul Roberts proved the new intersection theorem in general (i.e., mixed characteristic) [66, 67], using the theory of local Chern classes in all characteristics developed by Fulton, MacPherson, and Baum [6], for which there is an excellent exposition in [29]. Ray Heitmann proved the direct summand conjecture in dim 3 [33], and I used his main result to show that big Cohen-Macaulay algebras exist in dimension 3 in mixed characteristic [43]. His results in [33, 34] show that if (R, m) is a complete local domain of dimension 3 and R^+ denotes its *absolute integral closure*, i.e., its integral closure in an algebraic closure of its fraction field, then every element of the maximal of R^+ kills $H^2_m(R^+)$.

The direct summand conjecture implies the canonical element conjecture [41], a very strong form of the new intersection which we refer to here as the *strong intersection theorem* (it has also been called the improved new intersection theorem), which is discussed in the next section, and many other results. The interactions of these various conjectures, all of which are now theorems, are studied in [41, 20].

3. The strong intersection theorem, the syzygy problem, and the generalized principal ideal theorem

In [41] I showed that the direct summand conjecture is equivalent to the canonical element conjecture, and that these imply the following statement, the strong intersection conjecture. All of these are now theorems. The strong intersection theorem is the following.

Theorem 3.1. *If* $(R, \mathfrak{m}, K)$ *is local and* $0 \to G_s \to \cdots \to G_0 \to 0$ *is a complex of finitely generated free modules such that* $H_0(G_\bullet)$ *has a minimal generator that is killed by a power of* $\mathfrak{m}$ *and* $H_i(G_\bullet)$ *has finite length for* $i \geq 1$*, then* $d = dim(R) \leq s$.

The strong intersection theorem was proved by Graham Evans and Phil Griffith in equal characteristic in their proof of the syzygy conjecture:

Theorem 3.2 (Evans-Griffith syzygy theorem). *Let* $(R, \mathfrak{m})$ *be a local catenary domain that contains a field. Let* M *be a non-free* S_k*-module of rank* r *and finite projective dimension. Then* r *is greater than or equal to* k*. In particular, over a regular local ring, every non-free* k *th module of syzygies has rank at least* k.

The characteristic restriction is only needed because they use the existence of big Cohen-Macaulay modules: these are used to prove the strong intersection conjecture. In [41] I showed that the direct summand conjecture implies the strong intersection conjecture. Hence, the theorem above holds whether R contains a field or not. Sankar Dutta proved that the strong intersection conjecture implies the direct summand conjecture: they are equivalent.

It is worth noting that Evans and Griffith comment in [27] that their result, Theorem 3.2 above, is a sort of converse to one of the beautiful results of Winfried Bruns in [13]. They referred specifically to Bruns's result that if M is a k th syzygy of rank $k + r$ with $r > 0$, then M has an R-free submodule G of rank r such that M/G is a k th syzygy of rank k.

We note that Eisenbud and Evans proved the following result in equal characteristic using the existence of big Cohen-Macaulay modules: the idea used is strongly related to the argument in [27], and only depends on knowing the strong intersection theorem.

Theorem 3.3 (generalized principal ideal theorem). *Let* R *be a noetherian ring,* M *a finitely generated* R*-module, and* $x \in M$*. If there is a prime ideal* $\mathfrak{p}$ *of* R *with* $x \in \mathfrak{p}M_\mathfrak{p}$*, and* I *is the ideal of* $\{f(x) : f \in \mathrm{Hom}_R(M, R)\}$*, then* *height*$(I) \leq \mathrm{rank}(M)$.

Bruns gave a beautiful proof in [14] that Theorem 3.3 holds without the restriction to equal characteristic. Of course, the fact that the direct summand conjecture is now known also gives a proof of the general case, but the argument is much more difficult than the one in [14] when one considers the underlying machinery.

4. Serre multiplicities

Let $(T, \mathfrak{n}, K)$ be a regular local ring of dimension n and let M, N be nonzero finitely generated modules such that $\ell(M \otimes_T N) < \infty$, i.e., $\mathrm{Supp}(M) \cap \mathrm{Supp}(N) = \{\mathfrak{m}\}$. Here, $\ell(\ )$ denotes length. Serre's intersection multiplicity [69] is defined by the formula

$$\chi^T(M, N) = \chi(M, N) := \sum_{i=0}^{n} (-1)^i \ell\big(\mathrm{Tor}_i^T(M, N)\big).$$

$\chi(M, N)$ is bi-additive in M and N when it is defined on all the pairs occurring. Since M and N have finite filtrations in which all factors are prime cyclic modules T/P, the behavior of χ is determined by its behavior on pairs of such modules T/P, T/Q, where P, Q are prime and $P + Q$ is $\mathfrak{n}$-primary.

This is a formal situation analogous to studying the intersection of two varieties near an isolated point of intersection.

In equal characteristic, T is regular iff its completion is a formal power series ring over a field. In mixed characteristic, it may be formal power series over a complete DVR (V, pV) (like the p-adic numbers) whose maximal ideal is generated by the characteristic p of the residue class field.

But there is a frequently more difficult ramified case where the ring has the form

$$V[[X_1, \ldots, X_d]]/(p - F).$$

Here F is in the square of the maximal ideal. Such a ring, in general, is regular but *not* a formal power series ring over a DVR. For a specific example of this type, consider $V[[X, Y, Z]]/(p - X^3 - Y^5 - Z^7)$.

Serre proved [69] that if T is regular local and its completion is formal power series over a field or a DVR, then the following hold for finitely generated nonzero modules M, N when $\ell(M \otimes_R N) < \infty$ (think of the case where $M = T/P$, $N = T/Q$):

(a) $\dim(M) + \dim(N) \leq \dim(T)$.
(b) $\dim(M) + \dim(N) < \dim(T) \Rightarrow \chi(M, N) = 0$.
(c) $\dim(M) + \dim(N) = \dim(T) \Rightarrow \chi(M, N) > 0$.

These results are highly non-trivial and very powerful. For example, part (a) implies that the height of a prime ideal in a regular ring cannot increase if it stays proper when you map to another ring, which greatly generalizes the well-known results of Eagon and Northcott on heights of determinantal ideals.

Serre also proved (a) $\dim(M) + \dim(N) \leq \dim(T)$ for any regular local ring T, and, essentially, conjectured ("Il est naturel de conjecturer") that (b) and (c) hold as well. The remaining case is the ramified case in mixed characteristic. It has been an open question for over fifty years. Serre also proved the case when either M or N is $T/(f_1, \ldots, f_h)T$ where $f_1, \ldots, f_h$ is a regular sequence, i.e. the case where one of the modules is a complete intersection.

Serre's results inspired a great deal of additional work on multiplicities defined by similar methods in this and even greater generality. We refer the reader to [19, 21, 22, 23, 24, 25, 26, 58] as well as to several other papers cited explicitly just below.

Paul Roberts and, independently, H. Gillet and C. Soulé, proved
(b) $\dim(M) + \dim(N) < \dim(T) \Rightarrow \chi(M, N) = 0$.

See [65] and [31].

O. Gabber, using de Jong's results on alterations [17], proved that $\chi(M,N) \geq 0$ in the ramified case. Gabber's argument is given in [7].

All that remains is to show that $\chi(M,N)$ is strictly positive in the ramified case. This remains open. One may assume that T is complete with a perfect (or even algebraically closed) residue field. Also, it suffices to prove the result when $M = T/P$ and $N = T/Q$ are prime cyclic modules.

An important point here is that $\chi(T/P, T/Q)$ is positive when T/P and T/Q both have small Cohen-Macaulay modules. Therefore, the existence of small Cohen-Macaulay modules implies the remaining case of Serre's conjecture. In fact, this idea can be used to settle Serre's conjecture on positivity of multiplcities up through dimension 4.

We give the argument. Keep in mind that $P + Q$ is $\mathfrak{n}$-primary, and that $\dim(T/P)+\dim(T/Q) = \dim(T)$. Suppose M is a small Cohen-Macaulay module for $R = T/P$ of (torsion-free) rank r and that N is a small Cohen-Macaulay module for $S = T/Q$ of rank s. M has a finite filtration with r factors equal to R and other factors of smaller dimension. N has a finite filtration with s factors equal to S and other factors of smaller dimension. Using the bi-additivity of χ and the fact that the vanishing part (b) of the conjecture is known, one obtains that $\chi(M,N) = rs\,\chi(R,S)$. But when M, N are Cohen-Macaulay the higher Tors vanish, and $\chi(M,N) = \ell(M \otimes_T N) > 0 \Rightarrow \chi(R,S) > 0$. □

5. Lim Cohen-Macaulay sequences of modules

One of the main points here is that the idea in the proof given at the end of the preceding section can be made to work with a much weaker assumption than the existence of small Cohen-Macaulay modules for the complete local domains R and S. We want to introduce the notion of a sequence of modules in which the terms are getting "closer and closer" to being Cohen-Macaulay, compared to their torsion-free ranks.

If $\underline{x} = x_1, \ldots, x_d \in R$ let $H_\bullet(\underline{x}; M)$ denote Koszul homology, and let $h_i(\underline{x}; M)$ denote its length when the length is finite. A sequence of nonzero finitely generated modules $\mathcal{M} = \{M_n\}_n$ of dimension $\dim(R)$ over a local domain $(R, \mathfrak{m}, K)$ is defined to be *lim Cohen-Macaulay* if for some (equivalently, every) system of parameters $\underline{x} = x_1, \ldots, x_d$, for $1 \leq i \leq d$ we have

$$h_i(\underline{x};\, M_n) = o\big(\nu(M_n)\big).$$

It is not obvious, but it is true that the condition is independent of the choice of parameters $\underline{x}$.

If R is a local domain, and $\{M_n\}_n$ is lim Cohen-Macaulay, then

$$1 \leq \frac{\nu(M_n)}{\mathrm{rank}(M_n)} \leq e(R) + \epsilon_n$$

where $e(R)$ = the multiplicity of R and $\epsilon_n \to 0$ as $n \to \infty$. Hence, $\{M_n\}_n$ is lim Cohen-Macaulay iff for some (equivalently, every) system of parameters $\underline{x}$, $h_i(\underline{x};\, M_n) = ø\big(\mathrm{rank}(M_n)\big), 1 \leq i \leq d$. Also, one can prove that $h_0(\underline{x}; M) = O\big(\mathrm{rank}(M_n)\big)$.

Note that if R has a small Cohen-Macaulay module M, we may take the sequence to be the constant sequence $M, M, M, \ldots, M, \ldots$.

If $(R, \mathfrak{m}, K)$ is a complete local domain of char. $p > 0$ with K perfect, R has a lim Cohen-Macaulay sequence! We note the following result from [11].

Theorem 5.1. *If* $(R, \mathfrak{m}, K)$ *is a complete local domain of char.* $p > 0$ *with* K *perfect, then* $M_n := R^{1/p^n}$ *is a lim Cohen-Macaulay sequence for* R.

This follows from standard results in tight closure theory, and also can be deduced from the results of [19]. The same holds if R is F-finite (i.e., the Frobenius map $F : R \to R$ is module-finite.) This is a vast improvement over what we know about existence of small Cohen-Macaulay modules.

We note the following two facts:

If $R \to S$ is local and module-finite, a sequence of nonzero S-modules is lim Cohen-Macaulay over S if and only if it is lim Cohen-Macaulay over R.

If R is a regular local, a sequence $\{M_n\}_n$ of R-modules is lim Cohen-Macaulay if and only if all higher Betti numbers of M_n are quite small, in the sense of $o(\_\_)$, compared to the 0 th Betti number, which is $\nu(M_n)$.

Conjecture 5.2. *Every complete local domain of mixed characteristic with algebraically closed residue class field has a lim Cohen-Macaulay sequence of modules.*

We shall see in the next section, Theorem 6.1 and Corollary 6.2, that this conjecture is sufficient to prove the Serre conjecture on positivity of multiplicities, and that it also implies the existence of big Cohen-Macaulay modules (hence, also, the direct summand conjecture) in mixed characteristic: see Theorem 8.2.

6. Lim Cohen-Macaulay sequences and the positivity of Serre multiplicities

We note the following result from [11]:

Theorem 6.1. *Let* P *and* Q *be primes of a complete regular local ring* $(T, \mathfrak{n}, K)$, *let* $R := T/P$ *and* $S := T/Q$, *and suppose that* $P + Q$ *is* $\mathfrak{n}$*-primary and that*

$$\dim(R) + \dim(S) = \dim(T).$$

If R *and* S *have lim Cohen-Macaulay sequences, say* $\{M_n\}_n$ *and* $\{N_n\}_n$, *resp., then* $\chi(R, S) > 0$.

$$\textit{In fact, } \ \chi(R, S) = \lim_{n\to\infty} \frac{\ell(M_n \otimes_T N_n)}{\mathrm{rank}(M_n)\mathrm{rank}(N_n)}.$$

All terms are $\geq \dfrac{\nu(M_n)}{\mathrm{rank}(M_n)} \dfrac{\nu(N_n)}{\mathrm{rank}(N_n)} \geq 1.$

Corollary 6.2. *If all complete local mixed characteristic domains with perfect (or even algebraically closed) residue class fields have lim Cohen-Macaulay sequences of modules, then Serre's conjecture on multiplicities is true in general.*

This case gives the general result by using standard reductions in the problem.

To give some idea of the proof of Theorem 6.1, we need to define multiple Tor over a ring T, i.e., Tor applied to a sequence of inputs, not just two. The inputs are left complexes of flat T-modules or simply T-modules.

If the input is a module, replace it by a projective (or flat) resolution: the choice won't matter. Multiple Tor is the homology of the total tensor product of these complexes. This agrees with the usual definition of $\mathrm{Tor}^T_\bullet(M, N)$ for the case of two modules. By viewing the total complex as a double complex in various ways, one gets a number of spectral sequences.

Theorem 6.1 follows if $\ell\big(\mathrm{Tor}_i^T(M_n, N_n)\big) = o\big(\mathrm{rank}(M_n)\mathrm{rank}(N_n)\big)$ for $i \geq 1$. One can choose a system of parameters $\underline{x}, \underline{y}$ for T such that the $x_i \in Q$ and their images form a system of parameters in $R = T/P$ vand the y_j are in P and their images form a system of parameters in $S = T/Q$. One uses a spectral sequence argument to show that $\Theta_n := \mathrm{Tor}^T_{i+d}(T/(\underline{x}, \underline{y}), M_n, N_n)$ maps onto $\mathrm{Tor}^T_i(M_n, N_n)$. $\Theta_n \cong \mathrm{Tor}^T_{i+d}\big(T/(\underline{x}), T/(\underline{y}), M_n, N_n\big)$ (quadruple Tor) $\cong \mathrm{Tor}^T_{i+d}(T/(\underline{x}), M_n, T/(\underline{y}), N_n)$.

By grouping together the first two inputs and the last two, one gets, using two spectral sequences, that the length of Θ_n is bounded by the the sum of the lengths of the $\mathrm{Tor}^T_r\big(H_s(\underline{x}; M_n), H_t(\underline{y}, N_n)\big)$ for $r+s+t = i+d$. If $r > d$ the Tor vanishes. If $r \leq d$ then $s + t \geq i > 0$. The respective inputs to Tor^T_r are at worst $O\big(\mathrm{rank}(M_n)\big)$ and $O\big(\mathrm{rank}(N_n)\big)$ and at least one of s, t is positive, so at least one of these can be replaced by o. It follows that Θ_n is $o\big(\mathrm{rank}(M_n)\mathrm{rank}(N_n)\big)$. □.

7. Lim Cohen-Macaulay sequences yield closure operations

We shall use lim Cohen-Macaulay sequences over a local ring $(R, \mathfrak{m}, K)$ to define a closure operation $\natural$ on submodules of finitely generated modules. The good properties of this closure operation enable one to show that if a local ring has a lim Cohen-Macaulay sequence, then it a big Cohen-Macaulay module. Moreover, one can get many other closure operations from sequences of finitely generated modules (or even nets of modules: the index set is a directed set, not necessarily the positive integers) over any local ring.

Let $\mathcal{M}$ denote $\{M_n\}_n$, a sequence of nonzero modules over a local ring $(R, \mathfrak{m}, K)$. Let α be a function from the set of modules $\{M_n : n \in \mathbb{N}\}$ to $\mathbb{N}$, the nonnegative integers. Our main interest is in the cases where α is ν (number of generators) or torsion-free rank. For each such α we define a closure operation on submodules of finitely generated R-modules, which we refer to as $\mathcal{M}$-closure with respect to α. If $A \subseteq B$ are R-modules, we use the notation $A^\natural_B$ for the $\mathcal{M}$-closure of A in B with respect to α.

These closure operations on submodules A of finitely generated R-modules B are defined as follows. If B/A has finite length, we define the closure $A^\natural_B$ to be the largest submodule A' of B containing A such that

$$(\dagger_\alpha) \quad \ell\big(\mathrm{Im}(M_n \otimes_R A'/A \to M_n \otimes_R B/A)\big) = o\big(\alpha(M_n)\big).$$

In general, we define the $\mathcal{M}$-closure of A in B with respect to α as

$$\bigcap_{t=1}^{\infty} (A + \mathfrak{m}^t B)^\natural_B.$$

This gives the same result as the original definition if B/A has finite length.

The subscript $_B$ is often omitted if B is clear from context. In discussing ideals of R, $I^\natural$ is always $I^\natural_R$.

Example. If R is a local Noetherian domain, consider the net of all nonzero ideals, where $I \leq J$ means that $J = II'$ for some nonzero ideal I'. The closure with respect to rank is integral closure.

Theorem 7.1. *Let $(R, \mathfrak{m}, K)$ be a local domain and let $\mathcal{M} = \{M_n\}_n$ be a sequence of finitely generated R-modules that are not torsion. Suppose that $\widehat{R}$ is equidimensional, which is true if R is excellent. Then for every ideal I of R, the closure of I with respect to $\mathcal{M}$ and rank is contained in $\overline{I}$, the integral closure of I.*

Equivalently, integrally closed ideals of I are $\mathcal{M}$-closed with respect to rank. Hence, radical ideals are $\mathcal{M}$-closed with respect to rank, and so prime ideals are $\mathcal{M}$-closed with respect to rank. In particular, 0 and $\mathfrak{m}$ are $\mathcal{M}$-closed in R with respect to rank.

From now on we discuss only closures coming from lim Cohen-Macaulay sequences with respect to ν. If the ring is a domain, this is the same as closure with respect to rank. Here is one rather trivial example.

Example. If R is Cohen-Macaulay, the closure coming from the lim Cohen-Macaulay sequence R, R, R, $\ldots$ and ν (or rank) is the identity closure. (Every submodule of every module is closed.)

The lim Cohen-Macaulay sequences discussed in Theorem 5.1 give tight closure:

Theorem 7.2. *If R is a local domain of characteristic $p > 0$, then the closure operartion given by the lim Cohen-Macaulay sequence $\{R^{1/p^n}\}_n$ (i.e., $\{F^n_*(R)\}_n$) is tight closure.*

This can be deduced from length characterizations of tight closure in §8 of of [44].

All lim Cohen-Macaulay closures $\natural$ over a complete local domain R satisfy colon-capturing: if $x_1, \ldots, x_d$ is a system of parameters then if $I = (x_1, \ldots, x_k)R$, $I :_R x_{k+1} \subseteq I^\natural$. Better: $I^\natural_R : Rx_{k+1} \subseteq I^\natural$.

Also, if $s \geq 1$, $(x_1, \ldots, x_{k-1}, x_k^{s+1})^\natural :_R x_k \subseteq (x_1, x_2, \ldots, x_{k-1}, x_k^s)^\natural$.

Moreover, lim Cohen-Macaulay closures satisfy the somewhat mysterious Dietz colon-capturing axiom from [18]:

(7) Let $x_1, \ldots, x_{k+1}$ be part of a system of parameters for R, and let $J = (x_1, \ldots, x_k)R$. Suppose that one has a surjection $f : B \twoheadrightarrow R/J$ and $v \in B$ such that $f(v) = x_{k+1} + J \in R/J$. Then $(Rv)^\natural_B \cap \mathrm{Ker}(f) \subseteq (Jv)^\natural_B$.

8. The Dietz axioms.

Let $(R, \mathfrak{m})$ be a fixed complete local domain. Let $\natural$ denote a closure operation over R chart assigns to every R-submodule A of a finitely generated R-module B a submodule $A^\natural_B$ of B. Let A, B, and C be arbitrary finitely generated R-modules with $A \subseteq B$. By the *Dietz axioms* we mean the following seven conditions on $\natural$.

(1) $A^\natural_B$ is a submodule of B containing A.
(2) $(A^\natural_B)^\natural_B = A^\natural_B$.
(3) If $A \subseteq B \subseteq C$, then $A^\natural_C \subseteq B^\natural_C$.
(4) Let $f : B \to C$ be an R-linear homomorphism. Then $f(A^\natural_B) \subseteq f(A)^\natural_C$.
(5) If $A^\natural_B = A$, then $0^\natural_{B/A} = 0$.
(6) The ideals $\mathfrak{m}$ and 0 are $\natural$-closed in R.
(7) Let $x_1, \ldots, x_{k+1}$ be part of a system of parameters for R, and let $J = (x_1, \ldots, x_k)R$. Suppose that one has a surjective map $f : B \twoheadrightarrow R/J$ and $v \in B$ such that $f(v) = x_{k+1} + J \in R/J$. Then $(Rv)^\natural_B \cap \mathrm{Ker}(f) \subseteq (Jv)^\natural_B$.

Theorem 8.1 (Dietz). *A complete local domain R has a big Cohen-Macaulay module if and only if it has a closure operation on submodules of finitely generated modules satisfying axioms (1)–(7) above.*

In [11] it is proved that the closure coming from a lim Cohen-Macaulay sequence of modules is a Dietz closure, i.e., it satisfies the Dietz axioms. Hence:

Theorem 8.2. *If R is a local ring that has a lim Cohen-Macaulay sequence of modules, then R has a big Cohen-Macaulay module.*

Of course, this also means that proving the existence of lim Cohen-Macaulay sequences of modules in mixed characteristic would give a new proof of the direct summand conjecture.

9. Lech's conjecture

Over fifty years ago, Lech made the following:

Conjecture 9.1 (Lech). *If $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is a flat local extension of local rings, then $e(R) \le e(S)$.*

The case where $\dim(R) = \dim(S)$ and R, S are complete implies the general case. Lech proved the result in dimension at most 2. Recently, Linquan Ma [55] used the fact that R^{1/p^n} is a lim Cohen-Macaulay sequence as well as other ideas to prove the case where R is of dimension 3 and characteristic $p > 0$, and this implies the case where R has dimension 3 and contains a field, by reduction to characteristic $p > 0$.

10. The direct summand conjecture and Q-sequences

Let M be a finitely generated R-module killed by a power of an ideal I of a ring R. We define the *quasilength* $\mathcal{L}_I(M)$ of M with respect I to be the length of a shortest finite filtration of M in which each factor is a homomorphic image of R/I, i.e., a cyclic R-module annihilated by I. This notion is introduced and studied in [48], and the theory is developed further in [52].

If I is an ideal of R generated by d elements, say $\underline{x} = x_1, \dots, x_d$, let $I_t = (x_1^t, \dots, x_d^t)R$. The *content* of a dth local cohomology module $H^d_I(M)$ with respect to a specific sequence of d generators $\underline{x} = x_1, \dots, x_d$ of I is defined as

$$\lim_{t\to\infty} \frac{\mathcal{L}_I(M/I_tM)}{t^d},$$

if it exists. Evidently, the order of the elements in the sequence does not matter.

We are primarily interested here in the case where $M = R$. Since R/I_t always has a filtration by t^d monomial ideals in which the factors are homomorphic images of R/I, the content of $H^d_I(R)$, if it exists, is always in the unit interval $[0, 1]$. It is shown in [48] that whether the content of the local cohomology is 1 is independent of the choice of d generators for I. In this case, the local cohomology is called *robust*, and the elements $x_1, \dots, x_d$ are called a *Q-sequence* in R. Alternatively, we may simply say that $\underline{x}$ is a *Q-sequence* if for all positive integers t, every finite filtration of R/I_t with factors that are cyclic modules that are homomorphic images of R/I has at least t^d terms.

A regular sequence in a Noetherian ring is a Q-sequence. Surprisingly, it is not known whether a regular sequence in a ring that is not Noetherian must be a Q-sequence, even in the case where d = 2. In [48], it is shown that a system of parameters for an equicharacteristic local ring is a Q-sequence. This is not known in mixed characteristic, even in dimension 3, although it is known in mixed characteristic that the content of the dth local cohomology of a local ring of dimension d with respect to a system of parameters exists and is positive.

Conjecture 10.1. *Every system of parameters of every local ring is a Q-sequence.*

It is shown in [48] that this conjecture implies the direct summand conjecture. Conjecture 10.1 remains open.

In [52], the notion of a *robust* algebra over a local ring R is studied: S is defined to be *robust* if every system of parameters for R is a Q-sequence in S. This leads to a notion of closure for ideals: J is closed in the sense of robust closure if whenever an element $u \in R$ is such that $u \in JS$ for some robust R-algebra S, the element u is in J. The robust closure of an ideal is defined to be the smallest ideal containing it that is closed in the sense of robust closure. It is shown in [48] that this closure agrees with tight closure for complete local domains of characteristic p. It remains an open question whether this notion gives a valuable generalization of tight closure in the mixed characteristic case, since we do not even know that a complete local domain is robust as an algebra over itself: this would follow from Conjecture 10.1.

Note that a number of other closures that might play, in mixed characteristic, a role similar to the one played by tight closure in equal characteristic are studied in [39, 41] and [50].

11. A variant notion of lim Cohen-Macaulay sequences

Let $e_I(M)$ denote the multiplicity of M with respect to the m-primary ideal I. A finitely generated R-module of dimension $d = \dim(R)$ is Cohen-Macaulay if and only if for one (equivalently, every) system of parameters $\underline{x} = x_1, \ldots, x_d$ or R we have

$$\ell\big(M/(\underline{x})M\big) = e_{(\underline{x})}(M).$$

.

The definition we have given for a lim Cohen-Macaulay sequence implies that for any given system of parameters $\underline{x} = x_1, \ldots, x_d$ for the local ring R,

$$(\dagger) \qquad \lim_{n\to\infty} \frac{\ell(M_n/(\underline{x})M_n)}{e_{(\underline{x})}(M_n)} = 1.$$

When one has a sequence of modules of dimension equal to that of R that satisfies this weaker condition for a system of parameters $\underline{x}$, one can show that the monomial conjecture holds for these parameters. That is, for all positive integers t, one has

$$(*) \qquad (x_1 \cdots x_d)^{t-1} \notin (x_1^t, \ldots, x_d^t)R.$$

To see why, let $I_s = (x_1^s, \ldots, x_d^s)R$. Note that contradicting $(*)$ gives a filtration of R/I_t, constructed using ideals generated by monomials, that has length $t^d - 1$ instead of t^d. The fact that

$$(x_1 \cdots x_d)^{t-1} \in I_t$$

enables one to drop one factor from the obvious filtration. Each factor is a homomorphic image of R/I, where $I = I_1 = (x_1, \ldots, x_d)R$.

This implies that for every s, R/I_{ts} has a filtration by $(t^d-1)s^d$ monomial ideals, where each factor is a homomorphic image of R/I.

One first filters with s^d homomorphic images of R/I_t, and then filters each of these as above. Expanding the ideals to M_n, we obtain a filtration of $M_n/I_{ts}M_n$ by $(t^d-1)s^d$ homomorphic images of M_n/IM_n.

This shows that

$$\ell(M_n/I_{st}M_n) \le (t^d-1)s^d\ell(M_n/IM_n)$$

or

$$\frac{\ell(M_n/I_{st}M_n)/(st)^d}{\ell(M_n/IM_n} \le \frac{(t^d-1)s^d}{t^ds^d} = 1 - \frac{1}{t^d}.$$

As $n \to \infty$ the numerator on the left has limit $e_{(\underline{x})}(M_n)$. If we now let $n \to \infty$, we obtain a contradiction. □

This argument is a variant of one given in [48] proving that the content of local cohomology of a local ring with respect to a system of parameters is positive: see §10.

Even if $R = K[[x,y]]$, one can let $M_n := R^n \oplus K$, where $K = R/(x,\, y)$, which gives a lim Cohen-Macaulay sequence in the original (i.e., strong) sense, or one can let $M_n := R^n + R/m^{n^n}$, which gives a lim Cohen-Macaulay sequence in the weak sense (the multiplicity is n, the length of $M_n/(x,y)M_n$ is $n+1$, but $H_2(x,y;M_n) \cong m^{n^n-1}/m^{n^n}$ is enormous compared to $\nu(M_n) = n+1$.

The following is, evidently, a weakening of Conjecture 5.2

Conjecture 11.1. *A complete local domain with algebraically closed residue class field has a sequence of nonzero finitely generated modules such that*

$$\lim_{n\to\infty} \frac{\ell(M_n/(\underline{x})M_n)}{e_{(\underline{x})}(M_n)} = 1.$$

12. A local cohomology conjecture and a tight closure conjecture

We recall that the *absolute integral closure* R^+ of a domain R is the integral closure of R in an algebraic closure of its fraction field. If R is a complete local domain of characteristic $p > 0$, then R^+ is a big Cohen-Macaulay algebra [45]. In mixed characteristic p, Heitmann [33] first proved that if $(R, \mathfrak{m})$ is a mixed characteristic p complete local domain of dimension 3, then for all n, $p^{1/n}$ kills $H^2_{\mathfrak{m}}(R^+)$, although the result is not stated in precisely this way in [45], and that this implies the direct summand conjeccture. Heitmann's result was used in [42] to prove the existence of big Cohen-Macaulay algebras in dimension 3. In [34], Heitmann proved that the entire maximal ideal of R^+ kills $H^2_{\mathfrak{m}}(R^+)$ for mixed characteristic complete local domains $(R,\, \mathfrak{m})$ of dimension 3.

The following two conjectures are open: a proof of either would would give a new proof of the direct summand conjecture.

Conjecture 12.1. *Let $(R,\, \mathfrak{m})$ be a complete local domain of mixed characteristic p and Krull dimension d. Then the maximal ideal of R^+ kills $H^{d-1}_{\mathfrak{m}}(R^+)$.*

In [52] it is shown that Conjecture 12.1 implies:

Conjecture 12.2. *Let R be a complete local domain of mixed characteristic p. Suppose that $x_1,\, \dots,\, x_{d-1},\, x_d = p$ is a system of parameters for R, and let t be a positive integer. Then the image of $(x_1,\, \dots,\, x_{d-1})R :_R p^t$ in $\overline{R} = R/pR$ is contained in the tight closure of $(x_1,\, \dots,\, x_{d-1})\overline{R}$*

This results of [33, 34] and [52] show that both these conjectures are true if $d = 3$ (or is smaller).

For background on tight closure theory, we refer the reader to [15, 44, 46], and [53].

The initial motivation for Cojecture 12.2 came from the much stronger conjecture that there is a notion of tight closure for all local rings that agrees with the usual notion for local rings of positive characteristic, has the colon-capturing property in complete domains, and is persistent.

13. Ranganathan's strong direct summand conjecture and the vanishing conjecture for maps of Tor

The strong direct summand conjecture, studied by N. Ranganathan in [59], remains open.

Conjecture 13.1 (Ranganathan's strong direct summand conjecture). *For $A \subseteq R$, suppose R is a module-finite domain over A, where A is regular local. Let x be a regular parameter (i.e., that $x \in \mathfrak{m}_A - \mathfrak{m}_A^2$) and let q be a height 1 prime in R lying over xA. Then xA is a direct summand of q as an A-module.*

It is shown in [59] that this is equivalent to the vanishing conjecture for maps of Tor:

Conjecture 13.2 (vanishing conjecture for maps of Tor). *Let $A \to R \to S$ be Noetherian rings, where A is a regular domain, S is module-finite and torsion-free over A, and S is regular. Then for every A-module M and integer $i \geq 1$, the map $\mathrm{Tor}_i^A(M, R) \to \mathrm{Tor}_i^A(M, S)$ is 0.*

While the direct summand conjecture has now been settled, then strong form remains open, and the vanishing conjecture for maps of Tor remains open as well. There is a discussion of the vanishing conjecture for maps of Tor in [47], §4, where it is shown to follow from the weakly functorial existence of big Cohen-Macaulay algebras. In equal characteristic, it can be proved using tight closure theory. The vanishing conjecture for maps of Tor implies that if a Noetherian ring R is a direct summand, as an R-module, of a regular Noetherian ring S, then R is Cohen-Macaulay. Cf. [49, 12]. In [12] it is shown in the affine case that in characteristic 0 R has rational singularities. It appears that the recent results of André may be used to prove that direct summands of regular rings are Cohen-Macaulay in general.

Since [59] is not published, we give the proof of the equivalence of Conjecture 13.1 and Conjecture 13.2 here. We make use of several preliminary results, given just below.

From [1] we have:

Theorem 13.3. *If $A \to T$ is a local homomorphism of complete Noetherian local rings, and A is regular, then there is a complete regular local ring A' and a factorization $A \to A' \twoheadrightarrow T$, where the maps are local homomorphisms, A' is faithfully flat over A, and $A' \twoheadrightarrow T$ is surjective.*

We use the following fact [39] which holds under a mild condition (*approximately Gorenstein*) on the local ring A , and, in particular, holds when A is regular.

Theorem 13.4. *If A is regular local and W is a finitely generated A-module, then an injection $A \to W$ sending $1 \mapsto w$ splits if and only if for every ideal I of A, $IW \cap Aw = Iw$.*

Remark 13.5. In fact, the results of [39] show that when A is regular local or even Gorenstein local the map splits if for one system of parameters $x_1, \ldots, x_d$ for A and every $I_t = (x_1^t, \ldots, x_d^t)$ with $t \geq 1$, one has $I_tW \cap Aw = I_tw$.

The direct summand conjecture has now been proved, but that proof depends on a great deal of machinery, and we do not need to assume it in establishing the equivalence between the two conjectures, because each of the conjectures implies it. For Conjecture 13.2, this is proved in [47], (4.4), p. 81. For the strong direct summand conjecture, if $A \to S$ is a module-finite extension of a local ring, we may kill a minimal prime of S to reduce to the case where S is a domain, choose a regular parameter $x \in \mathfrak{m}_A - \mathfrak{m}_A^2$, and choose a minimal prime Q of S lying over xA. Since xA is a direct summand of Q, it is a direct summand of $xS \subseteq Q$, and this shows A is a direct summand of S.

We now state and prove the third preliminary result.

Lemma 13.6. *Let $A \hookrightarrow R \twoheadrightarrow T$ be maps of Noetherian rings, where A is a local domain, R is a domain, $A \hookrightarrow R$ is a module-finite extension, T is regular, and $Q = \mathrm{Ker}(R \twoheadrightarrow T)$ is a prime ideal of R. Assume that A is a direct summand of R. Assume also that $Q \cap A = P$ is the prime ideal of A generated by part of a regular system of parameters $x_1, \ldots, x_h$ (i.e., part of a minimal set of generators of the maximal ideal $\mathfrak{m}_A$), and $A/xA \cong R/Q \cong T$. Then for every ideal I of A, the map $\theta_I : \mathrm{Tor}_1^A(A/I, R) \to \mathrm{Tor}_1^A(A/I, T)$ is 0 if and only if $IQ \cap P = IP$.*

Hence, in the case where $h = 1$ and $x = x_1$, xA is a direct summand of Q if and only if all of the maps θ_I are 0.

Proof. We may tensor the short exact sequence $0 \to Q \to R \to T \to 0$ with A/I over A to obtain

$$\cdots \to \mathrm{Tor}_1^A(A/I, R) \xrightarrow{\theta_I} \mathrm{Tor}_1^A(A/I, T) \to Q/IQ \cdots$$

and so $\theta_I = 0$ if and only if

$$\mathrm{Tor}_1(A/I, T) \cong \mathrm{Tor}_1(A/I, A/P) \cong (I \cap P)/IP$$

injects into Q/IQ, i.e., if and only if $(I \cap P) \cap IQ \subseteq PI$. (Note that IQ is not necessarily contained in I here, because Q is an ideal of R.) Now $IQ \cap P \subseteq IR \cap A = I$, since $A \to R$ splits, so that I may be omitted from the triple intersection, and, hence, the vanishing of θ_I is equivalent to to the condition that $P \cap IQ \subseteq IP$. Since the opposite inclusion is automatic, we have proved the equivalence stated in the first paragraph.

The final statement follows at once from Theorem 13.4. □

The final statement in Lemma 13.6 yields at once that the vanishing conjecture for maps of Tor implies the strong direct summand conjecture: in fact, we only need the vanishing conjecture under the specific hypotheses of Lemma 13.6 in the case where $h = 1$, and we can assume the usual direct summand conjecture (so that we know A is a direct summand of R).

From Lemma 13.6 in the case where $h = 1$ we immediately have a global version:

Corollary 13.7. *Let $A \hookrightarrow R \twoheadrightarrow T$ be maps of Noetherian rings, where A, R are domains, $A \hookrightarrow R$ is a module-finite extension, T is regular, and $Q = \mathrm{Ker}(R \twoheadrightarrow T)$ is a height one prime of R. Assume that A is a direct summand of R. Assume also that $Q \cap A$ is a principal prime $P = xA$ such that A/xA is regular. Then xA splits from Q if and only for all ideals I of A, $xA \cap IQ \subseteq Ix$.*

Proof. Both statements are local on the maximal ideals of A. If x becomes a unit after localization, xA becomes A, Q becomes R, and the result is clear. If x does not become a unit, A is local and the conclusion follows from the final statement in Lemma 13.6. □

The final preliminary result is:

Lemma 13.8. *Let A be a regular ring, let P be a prime of A such that A/P is regular, let R be a module-finite extension ring of A, and let Q be a prime ideal of R lying over P. Then for every positive integer $n \in \mathbb{N}$, $P^nQ \cap A = P^{n+1}$.*

Proof. The issue is local on the maximal ideals of A. Hence, we may assume that A is local and then we know that P is generated by part of a regular system of parameters for A, and that P^{n+1} is primary to P. Suppose that $P^nQ \cap A$ contains an element b that is not P^{n+1}. Then this preserved when we local at $A - P$, and so we may assume that P is the maximal ideal of the regular local ring A. Moreover, the counterexample is preserved when we complete with respect to P, and also when we kill a minimal prime of R lying over (0) in A. Thus, we may assume that $A \subseteq R$ are complete domains, and, hence, that R is local. The P-adic valuation on A extends to a discrete rank one valuation v on R (with values in a cyclic subgroup of $\mathbb{Q}$) whose values on elements of Q are positive. This shows that the order of b with respect to v is greater than n, and since $b \in A$, $v(a)$ must be at least $n+1$. □

It remains only to show that the strong direct summand conjecture implies the vanishing conjecture for maps of Tor. Henceforth, we assume that the strong direct summand conjecture is true, and prove the vanishing conjecture for maps of Tor. We make several reductions in the problem.

Consider $A \subseteq R \to T$, where R is module-finite and torsion-free over the regular ring A, and let M be any A-module. We must show that for $i \geq 1$, the map $\mathrm{Tor}_i^A(M, R) \to \mathrm{Tor}_i^A(M, S)$ is 0. Since M is a direct limit of finitely generated A-modules, we may assume that M is finitely generated. By replacing M by a first module of syzygies of M repeatedly, we may assume that $i = 1$.

There is a presentation of M as the quotient of a free A-module:

$$M = \frac{Au_1 + \cdots + Au_s}{\mathrm{Span}\{\sum_{j=1}^s a_{ij}u_j : 1 \leq i \leq s\}}$$

By replacing A by $A[u] := A[u_1, \ldots, u_s]$, R by $R[u]$, T by $T[u]$, and M by the symmetric algebra of M over A, $\mathrm{Sym}_A(M) \cong A[u]/(\sum_{j=1}^s a_{ij}u_j : 1 \leq i \leq s)$, we may assume that M is a cyclic A-module A/I. Because a free resolution $G_\bullet$ of $\mathrm{Sym}_A(M)$ by free $A[u]$-modules is also a free resolution over A and $G_\bullet \otimes_{A[u]} R[u] \cong G_\bullet \otimes_A R$ (we may think of $R[u]$ as $A[u] \otimes_A R$), we may identify

$$\mathrm{Tor}_1^{A[u]}\big(\mathrm{Sym}_A(M),\, R[u]\big) \cong \mathrm{Tor}_1^A(\mathrm{Sym}_A(M), R)$$

and, similarly,

$$\mathrm{Tor}_1^{A[u]}\big(\mathrm{Sym}_A(M),\, T[u]\big) \cong \mathrm{Tor}_1^A(\mathrm{Sym}_A(M), T.$$

If the map between the Tor modules involving $\mathrm{Sym}_A(M)$ is 0, this is also true for M, because M is a direct summand of $\mathrm{Sym}_A(M)$ as A-modules.

In this way, we reduce to the case where M is a cyclic A-module, A/I. The issue of whether the map is 0 is local on T. We may therefore replace T by a localization, and complete T. We may likewise localize and complete A at the contraction of the maximal ideal of T to A. Since R is torsion-free over A, it is embeddable in a free A-module, and so remains torsion-free when we complete. The map $R \to T$ has a prime ideal as its kernel, and this prime contains a minimal prime of R lying over 0 in A. We may replace R by its quotient by this minimal prime. Hence, we may also assume that R is a complete local domain.

Now let A' be as in Theorem 13.3. By replacing A, M, and R by A', $M \otimes_A A'$ and $R \times_A A'$ respectively, we may assume that the map from A to T is a surjection, with kernel a prime ideal P. Note also that P must be generated by part of a regular system of parameters for A, since $A/P \cong T$ is regular. Let $P = (y_1, \ldots, y_h)$, where the y_i are part of a regular system of parameters for A and h is the height of P. We may again reduce to the case where R is a domain. We are still in the situation where M is a cyclic A-module, say A/I. Since A maps onto T, R also maps onto T. Let Q be the kernel of this map. Note that $Q \cap A = P$.

We are now in the situation of Lemma 13.6, with $M = A/I$. Hence, to complete the proof, it suffices to show that for all ideals I of A, $IQ \cap P = IP$. We shall achieve this by using an extended Rees ring to reduce to the case where $h = 1$.

Let t be a new indeterminate over R, let $v := 1/t$, and let $\widetilde{A}$ be the extended Rees ring of P, which is $A[Pt,\, v] = A[y_1t, \ldots, y_ht, v] \subseteq A[t, 1/t]$. Note that $\widetilde{A}$ is regular, since localizing at v yields $A[t, 1/t]$, while killing $v\widetilde{A}$ yields the associated graded ring of A with respect to $(y_1, \ldots, y_h)A$, which is a polynomial ring in h variables z_i over $T \cong A/P$. This, $v\widetilde{A}$ is a principal prime ideal of $\widetilde{A}$ such that the quotient is regular.

Let $\widetilde{R} := R[Pt, v]$. Evidently, this domain is module-finite over the regular domain $\widetilde{A}$. We want to show that there is a homogeneous prime ideal $\widetilde{Q}$ of $\widetilde{R}$ that contains Q and contracts to $v\widetilde{A}$. Let $J = Q\widetilde{R} + v\widetilde{R}$. Then

$$J = \cdots + Qv^n + \cdots + Qv + Q + QPt + QP^2t^2 + \cdots + QP^st^s + \cdots .$$

By Lemma 13.8, we have that $QP^s \cap P^s = P^{s+1}$. Hence, $J \cap \widetilde{A} = vA$. It follows that there is a prime $\widetilde{Q}$ of $\widetilde{R}$ containing J and lying over vA, and since it may be taken to be a minimal prime of J, it is homogeneous. Because we are assuming the strong direct summand conjecture, we have that $v\widetilde{A} \hookrightarrow \widetilde{Q}$ splits. For every ideal I of A, we then have that

$$(\dagger) \quad I\widetilde{Q} \cap v\widetilde{A} = Iv\widetilde{A}.$$

The degree 0 part of $\widetilde{Q}$ must be Q, since it contains Q and contracts to P in A. Comparing degree 0 components in ($\dagger$), we obtain that $IQ \cap P = IP$, as required. Hence, the strong direct summand conjecture implies the vanishing conjecture for maps of Tor. □

14. Big Cohen-Macaulay algebras, tight closure in all characteristics, and a recap of all the conjectures

In this section we collect all of the conjectures made so far, and also an additional conjecture on a very strong form of existence of big Cohen-Macaulay algebras. The conjectures below, most of which have been described earlier, are all open, and each of them implies the direct summand theorem. Three of the statements below are now theorems, and that is indicated. A diagram at the end shows what is known about the implications among them.

Conjecture 14.0 (Positivity of Serre intersection multiplicities). *If M, $N \neq 0$ are finitely generated modules over a regular local ring T such that $\ell(M \otimes_T N)$ is finite and $\dim(M) + \dim(N) = \dim(T)$, then the Serre intersection multiplicity $\chi(M, N) > 0$.*

The following three statements are now known, and we have used letters for them instead of numbers.

Theorem DS (Direct summand theorem). *A regular ring is a direct summand as a module over itself of every module-finite extension.*

Theorem CMA (Existence of big Cohen-Macaulay algebras). *Every local ring has a big Cohen-Macaulay algebra.*

Theorem CMM (Existence of big Cohen-Macaulay modules). *Every local ring has a big Cohen-Macaulay module.*

The remaining statements here are open. They all imply Theorem DS.

Conjecture 14.1 (Existence of lim Cohen-Macaulay sequences in a weak sense). *If R is a complete local domain of mixed characteristic with algebraically closed residue class field, then R has a sequence of modules M_n such that*

$$\lim_{n\to\infty} \frac{\ell(M_n/(\underline{x})M_n)}{e_{(\underline{x})}(M_n)} = 1.$$

See §11.

Conjecture 14.2 (Existence of lim Cohen-Macaulay sequences). *If R is a complete local domain of mixed characteristic with algebraically closed residue class field, then R has a lim Cohen-Macaulay sequence of modules.*

See §5.

Conjecture 14.3 (Systems of parameters are Q-sequences). *Every system of parameters of every local ring is a Q-sequence.*

See §10. For the next two conjectures, see §12.

Conjecture 14.4 (Capturing mixed characteristic parameter colon ideals with tight closure). *Let R be a complete local domain of mixed characteristic p. Suppose that $x_1, \dots, x_{d-1}, x_d = p$ is a system of parameters for R, and let t be a positive integer. Then the image of $(x_1, \dots, x_{d-1})R :_R p^t$ in $\overline{R} = R/pR$ is contained in the tight closure of $(x_1, \dots, x_{d-1})\overline{R}$*

Conjecture 14.5 (Next to top local cohomology is almost 0). *If $(R, \mathfrak{m})$ is a complete local domain of mixed chaacteristic of dimension d with algebraically closed residue class field, then $H^{d-1}_{\mathfrak{m}}(R^+)$ is killed by the maximal ideal of R^+.*

For the next two conjectures, see §13.

Conjecture 14.6 (Ranganathan's strong direct summand conjecture). *For $A \subseteq R$, suppose R is a module-finite domain over A, where A is regular local. Let x be a regular parameter (i.e., let x be in $\mathfrak{m}_A - \mathfrak{m}_A^2$) and let q be a height 1 prime in R lying over xA. Then xA is a direct summand of q as an A-module.*

Conjecture 14.7 (Vanishing conjecture for maps of Tor). *Let $A \to R \to S$ be Noetherian rings, where A is a regular domain, S is module-finite and torsion-free over A, and S is regular. Then for every A-module M and integer $i \geq 1$, the map $\mathrm{Tor}_i^A(M, R) \to \mathrm{Tor}_i^A(M, S)$ is 0.*

The last two conjectures have not been discussed separately.

Conjecture 14.8 (Existence of weakly functorial big Cohen-Macaulay algebras). *If $R \to S$ is a local map of complete local domains, there exists a map from a big Cohen-Macaulay algebra B over R to a big Cohen-Macaulay algebra C over S such that the diagram*

$$\begin{array}{ccc} B & \longrightarrow & C \\ \Big\uparrow & & \Big\uparrow \\ R & \longrightarrow & S \end{array}$$

commutes.

Conjecture 14.9 (Existence of weakly functorial big Cohen-Macaulay algebras, strong form). *One can assign a big Cohen-Macaulay algebra B_R to every complete local domain R such that B_R contains R^+ and given any local map $R \to S$ of complete local domains, there is a commutative diagram:*

$$\begin{array}{ccc} B_R & \longrightarrow & C_R \\ \Big\uparrow & & \Big\uparrow \\ R & \longrightarrow & S \end{array}$$

Conjecture 14.9, which is true in positive characteristic, would prove the existence of an analogue of tight closure in mixed characteristic with properties similar to those of tight closure in equal characteristic. If $N \subseteq M$ are finitely generated R-modules, one defines the closure $\mathrm{cl}_M(N)$ of N in M to be the kernel of the map $M \to B_R \otimes_R (M/N)$ such that $u \mapsto 1 \otimes (u + N)$. The characteristic p notions of tight closure (and plus closure) can be obtained in this way. For more details about closure operations induced by big Cohen-Macaulay modules and algebras, see [60, 61, 62].

With the recent breakthroughs, Conjecture14.9 may be in reach.

The following diagram, in which the arrows indicate implications, shows the relationships among these conjectures.

$$\begin{array}{ccccccccc}
 & & & & & & & & 14.0 \\
 & & & & & & & \nearrow & \\
 & & & & & & 14.2 & & \\
 & & & & & & \downarrow & \searrow & \\
 & & & & CMA & \to & CMM & & 14.1 \\
 & & & \nearrow & & & & \searrow & \downarrow \\
14.9 & \to & 14.8 & \to & 14.7 & \leftrightarrow & 14.6 & \to & DS \\
 & & & \searrow & & & & \nearrow & \uparrow \\
 & & & & 14.5 & \to & 14.4 & & 14.3
\end{array}$$

References

[1] L. Avramov, H.-B. Foxby, and B. Herzog, *Structure of local homomorphisms*, J. Algebra **164** (1994) 124–145.

[2] Y. André, *La lemme d'Abhyankar perfectoide*, preprint, 55 pp. arXiv:1609.00320 [math.AG] August 31, 2016.

[3] Y. André, *La conjecture du facteur direct*, preprint, 15 pp. arXiv:1609.00345 [math.AG] September 1, 2016.

[4] M. Auslander, *Modules over unramified regular local rings* in Proc. International Congress of Mathematics in Stockholm 1962, Uppsala, Almqvist and Wiksells, 1963, 230–234.

[5] H. Bass, *On the ubiquity of Gorenstein rings*, Math. Z. **82** (1963) 8–28.

[6] P. Baum, W. Fulton, and R. MacPherson, *Riemann-Roch for singular varieties*, Publ. Math. I.H.E.S. **45** (1975) 101–145.

[7] P. Berthelot, *Altérations de variétes algébriques [d'après A. J. de Jong]*, Séminaire Bourbaki, 48ème année, n$^{\text{O}}$ 815, pp. 815–01 – 815–39.

[8] B. Bhatt, *Derived splinters in positive characteristic*, Compos. Math. **148** (2012), 1757–1786.

[9] B. Bhatt, *Almost direct summands*, Nagoya Math. J. **214** (2014), 195–204.

[10] B. Bhatt, *On the direct summand conjecture and its derived variant*, preprint, 10 pp. arXiv:1608.08882v1[math.AG] 31 August, 2016.

[11] B. Bhatt, M. Hochster, and L. Ma, *Lim Cohen-Macaulay sequences of modules*, preliminary preprint, 38 pp., 2016.

[12] J.-F. Boutot, *Singularités rationelles et quotients par les groupes réductifs*, Invent. Math. **88** (1987) 65–68.

[13] W. Bruns, *"Jede" endliche freie Auflösung ist freie Auflösung eines von drei Elementen erzeugten Ideals*, J. Algebra **39** (1976) 429–439.

[14] W. Bruns, *The Eisenbud-Evans generalized principal ideal theorem and determinantal ideals*, Proc. of the Amer. Math. Soc. **83** (1981) 19–24.

[15] W. Bruns, *Tight closure*, Bull. Amer. Math. Soc. **33** (1996) 447–458.

[16] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge Studies in Advanced Mathematics **39**, 1993.

[17] A. J. de Jong, *Smoothness, semistability, and alterations*, Publ. Math. I.H.E.S. **83** (1996) 51–93.

[18] G. D. Dietz, *A characterization of closure operators that induce big Cohen-Macaulay modules*, Proc. Amer. Math. Soc. **138** (2010) 3849–3862.

[19] S. P. Dutta, *Frobenius and multiplicites*, J. Algebra **85** (1983) 424–448.

[20] S. P. Dutta, *On the canonical element conjecture*, Trans. Amer. Math. Soc. **299** (1987) 803–811.

[21] S. P. Dutta, *Generalized intersection multiplicities of modules*, Trans. Amer. Math. Soc. **276** (1983) 657–669.

[22] S. P. Dutta, *A special case of positivity*, Proc. Amer. Math. Soc. **103** (1988) 344–346.

[23] S. P. Dutta, *A note on Chow groups and intersection multiplicity of modules*, J. Algebra **161** (1993) 186–198.

[24] S. P. Dutta, *A theorem on smoothness–Bass-Quillen, Chow groups and intersection multiplicity of Serre*, Trans. Amer. Math. Soc. **352** (2000) 1635–1645.

[25] S. P. Dutta, *A special case of positivity. II.* Proc. Amer. Math. Soc. **133** (2005) 1891–1896.

[26] S. P. Dutta, M. Hochster, and J. E. McLaughlin, *Modules of finite projective dimension with negative intersection multiplicities*, Invent. Math. **79** (1985) 253–291.

[27] G. Evans and P. Griffith, *The syzygy problem*, Annals of Math. **114** (1981) 323–333.

[28] G. Faltings, *Almost étale extensions*, Ast'erisque **279** (2002)185–270.

[29] W. Fulton, *Intersection Theory*, Springer-Verlag, Berlin, 1984.

[30] O. Gabber and L. Ramero, *Almost ring theory*, Lecture Notes in Math. **1800**, Springer, 2003.

[31] H. Gillet and C. Soulé, *K-théorie et nullité des multiplicités d'intersection*, C. R. Acad. Sci. Paris Série I no. 3 t. **300** (1985) 71–74.

[32] R. C. Heitmann, *A counterexample to the rigidity conjecture for rings*, Bull. Amer. Math. Soc. New Series **29** (1993) 94–97.

[33] R. C. Heitmann, *The direct summand conjecture in dimension three*, Ann. of Math. (2) **156** (2002) 695–712.

[34] R. C. Heitmann, *Extensions of plus closure*, J. Algebra **238** (2001), 801–826.

[35] M. Hochster, *Cohen-Macaulay modules*, in Conference on Commutative Algebra, Lecture Notes in Math. No. **311** Springer-Verlag, Berlin and New York, 1973, 120–152.
[36] M. Hochster, *Contracted ideals from integral extensions of regular rings*, Nagoya Math. J. **51** (1973) 25–43.
[37] M. Hochster, *Topics in the homological theory of modules over commutative rings*, C.B.M.S. Regional Conf. Ser. in Math. No. **24**, Amer. Math. Soc., Providence, R.I., 1975.
[38] M. Hochster, *Big Cohen-Macaulay modules and algebras and embeddability in rings of Witt vectors*, Queen's Papers in Pure and Applied Math **42**, 1975, 106–195.
[39] M. Hochster, *Cyclic purity versus purity in excellent Noetherian rings*, Trans. Amer. Math. Soc. **231** (1977) 463–488.
[40] M. Hochster, *Parameter-like sequences and extensions of tight closure*, in Commutative Ring Theory and Applications, Lecture Notes in Pure and Applied Math. **231** Marcel Dekker, New York, 2003, pp. 267–287.
[41] M. Hochster, *Canonical elements in local cohomology modules and the direct summand conjecture*, J. of Algebra **84** (1983) 503–553.
[42] M. Hochster, *Solid closure* in *Commutative Algebra: Syzygies, Multiplicities and Birational Algebra*, Contemp. Math. **159**, Amer. Math. Soc., Providence, R. I., 1994, 103–172.
[43] M. Hochster, *Big Cohen-Macaulay algebras in dimension three via Heitmann's theorem*, J. Algebra **254** (2002) 395–408.
[44] M. Hochster and C. Huneke, *Tight closure, invariant theory, and the Briançon-Skoda theorem*, J. Amer. Math. Soc. **3** (1990) 31–116.
[45] M. Hochster and C. Huneke, *Infinite integral extensions and big Cohen-Macaulay algebras*, Annals of Math. **135** (1992) 53–89.
[46] M. Hochster and C. Huneke, *Tight closure of parameter ideals and splitting in module-finite extensions*, J. Algebraic Geom. **3** (1994) 599–670.
[47] M. Hochster and C. Huneke, *Applications of the existence of big Cohen-Macaulay algebras*, Advances in Math. **113** (1995) 45–117.
[48] M. Hochster and C. Huneke, *Quasilength, latent regular sequences, and content of local cohomology*, J. of Algebra **322** (2009) 3170–3193.
[49] M. Hochster and J. L. Roberts, *Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay*, Advances in Math., **13** (1974) 115–175.
[50] M. Hochster and J. Vélez, *Diamond closure*, in Algebra, Arithmetic and Geometry with Applications, Springer-Verlag, Berlin, 2004, 511–523.
[51] M. Hochster and Y. Yao, *Splitting theorems for the Frobenius endomorphism and strong F-regularity*, preprint.
[52] M. Hochster and W. Zhang, *Quasilength, content, and robust closure*, preprint.
[53] C. Huneke, *Tight Closure and Its Applications*, Proc. of the CBMS Conference held at Fargo, North Dakota, July, 1995, C.B.M.S. Regional Conference Series, Amer. Math. Soc., Providence, R.I., 1996.
[54] C. Lech, *Note on multiplicities of ideals* Ark. Mat. **4** (1960) 63–86.
[55] L. Ma, *Lech's conjecture in dimension three*, preprint, 26 pp., 2016.
[56] H. Matsumura, *Commutative Algebra*, Benjamin, New York, 1970.
J. Algebra **280** (2004) 505–536.
[57] C. Peskine and L. Szpiro, *Dimension projective finie et cohomologie locale*, Publ. Math. I.H.E.S. (Paris) **42** (1973) 323–395.
[58] C. Peskine and L. Szpiro, *Syzygies et multiplicités*, C. R. Acad. Sci. Paris Sér. A **278** (1974) 1421–1424.
[59] N. Ranganathan, *Splitting in module-finite extension rings and the vanishing conjecture for maps of Tor*, Thesis, University of Michigan, 2000.
[60] R. Rebhuhn-Glanz, *Closure operations that induce big Cohen-Macaulay modules and algebras, and classification of singularities*, Thesis, University of Michigan, 2016.
[61] R. Rebhuhn-Glanz, *Closure operations that induce big Cohen-Macaulay modules and classification of singularities*, preprint arXiv:1512.06843
[62] R. Rebhuhn-Glanz, *Closure operations that induce big Cohen-Macaulay algebras*, preprint, arXiv:1512.07862.
[63] P. Roberts, *Two applications of dualizing complexes over local rings*, Ann. Sci. Éc. Norm. Sup. **9** (1976) 103–106.

[64] P. Roberts, *Cohen-Macaulay complexes and an analytic proof of the new intersection conjecture*, J. Algebra **66** (1980) 225–230.
[65] P. Roberts, *The vanishing of intersection multiplicities of perfect complexes*, Bull. Amer. Math. Soc. **13** (1985) 127–130.
[66] P. Roberts, *Le théorème d'intersection*, C. R. Acad. Sc. Paris Sér. I **304** (1987) 177–180.
[67] P. Roberts, *Intersection theorems*, in *Commutative Algebra*, Math. Sci. Research Inst. Publ. **15** Springer-Verlag, New York-Berlin-Heidelberg,1989, 417–436.
[68] P. Scholze, *Perfectoid spaces*, Publ. Math. Inst. Hautes Études Sci. **116** (2012) 245–313.
[69] J.-P. Serre, Algèbre locale · Multiplicités, Springer-Verlag Lecture Notes in Math. No. 11, Séconde Edition, 1965.

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