TACKLING HOMOLOGICAL CONJECTURES USING ALGEBRAIC K-THEORY

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Abstract. This document represents notes for the presentation I will deliver at the *Homological Conjectures Workshop* at UIC on November 20, 2016.

Contents

1. Overview

The goal of this presentation is to illustrate how K-theory can be used, at least in some cases, to address problems in homological algebra. I will focus on two, related topics, one classical and one more modern: Serre's Vanshing Conjecture [Ser00] and a conjecture due to Dao-Kurano [DK14] on the stable Tor's of modules over hypersurface rings.

1.1. **Serre Vanshing Conjecture.** Let $Q = (Q, \mathfrak{m})$ be a (commutative Noetherian) local ring and M and N finitely generated Q-modules whose supports meet only at $\mathfrak{m}$; i.e., $\operatorname{supp}(M) \cap \operatorname{supp}(N) = \{\mathfrak{m}\}$. Then $\operatorname{Tor}_j^Q(M, N)$ has finite length for all $j \geq 0$. If we assume in addition that at least one of M and N has finite projective dimension, their *intersection multiplicity*, as defined by Serre, is the integer

$$\chi(M, N) = \sum_j (-1)^j \operatorname{length} \operatorname{Tor}_j^Q(M, N).$$

If Q is regular and $M = Q/\mathfrak{p}$ and $N = Q/\mathfrak{q}$ for prime ideals $\mathfrak{p}, \mathfrak{q}$, then χ should be interpreted as counting the multiplicity of the interesction of two subvarieties that meet only at a single point. This multiplicity should be behave in a vaguely topological sense: it ought to not vary under small perturbations.

For example, if $Q = k[[x, y, z]]$ and the heights of $\mathfrak{p}$ and $\mathfrak{q}$ are both two, then we are talking about two curves meeting at the origin in three space. A slight perturbation of either curve would cause them not to meet at all, and thus we would expect $\chi(Q/\mathfrak{p}, Q/\mathfrak{q}) = 0$, and indeed this holds true in this case. This intuition leads one to conjecture:

Conjecture 1.1 (Serre's Vanishing Conjecture (SVC))**.** *For a local ring Q, and finitely generated Q-modules M and N each of finite projective dimension, if* $\dim(M) + \dim(N) < \dim(Q)$ *then*

$$\chi(M, N) = 0.$$

In motto form: if M and N ought not to meet at all, then $\chi(M, N) = 0$.

Serre stated this conjecture when Q is regular, and he proved it when Q is regular and contains a field. Gillet-Soulé [GS87] and Roberts [Rob85] extended this to arbitrary regular local rings Q, working independently at about the same time in the middle 1980's. Both groups also prove it for complete intersection rings, and Roberts proves it for local rings with isolated singularities.

Remark 1.2. The statement of the conjecture makes sense if only one of M and N is assumed to have finite projective dimension. But in this case it is known to be false, due to an example by Hochster-Dutta-McLaughlin [DHM85].

In the first part of this talk, I will describe Gillet-Soulé's proof of the Serre Vanishing Conjecture. The version of this proof I present will involve a slight simplification that is due to some recent work [BMTW16b] by myself of my co-authors Brown, Miller and Thompson.

1.2. **A conjecture on the θ-pairing.** There is an analogue of the intersection multiplicity that is defined on the stable module category of a hypersurface ring that has only an isolated singularity.

By a *isolated hypersurface singulariy (i.h.s) ring* we shall mean a ring R isomorphic to $Q/(f)$ for a regular lcoal ring Q and non-zero element f such that $R_\mathfrak{p}$ is regular for all non-maximal primes $\mathfrak{p}$. For example,

$$R = k[[x, y, z, w]]/(xw - yz)$$

is an i.h.s. ring.

As with any hypersurface ring (isolated singularity or not), every finitely generated R-module has an eventually two periodic projective resolution. Let us illustrate this when $R = k[[x, y, z, w]]/(xw-yz)$ and $M = R/(x, z) \cong k[[y, z]]$. Then

$$0 \leftarrow M \leftarrow R \xleftarrow{(-\overline{z}, \overline{x})} R^2 \xleftarrow{\overline{A}} R^2 \xleftarrow{\overline{B}} R^2 \xleftarrow{\overline{A}} R^2 \xleftarrow{\overline{B}} R^2 \leftarrow \cdots$$

where

$$\overline{A} = \begin{bmatrix} \overline{x} & \overline{y} \\ \overline{z} & \overline{w} \end{bmatrix} \text{ and } \overline{B} = \begin{bmatrix} \overline{w} & -\overline{y} \\ -\overline{z} & \overline{x} \end{bmatrix}.$$

An important point is well-illustrated by this example: the "infinite tail" end of a such resolution is represented by a *matrix factorization*: a pair of square matrices A, B *with entries in* Q such that $AB = fI_n = BA$. In this case these matrices are

$$A = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \text{ and } B = A^{adj} = \begin{bmatrix} w & -y \\ -z & x \end{bmatrix}$$

Indeed, one can form a triangulated category $[mf(Q, f)]$ whose objects are such pairs of matrices and morphisms are, roughly speaking, homotopy classes of chain maps. Moreover, there is an equivalence

$$[mf(Q, f)] \cong D_{\text{sing}}(R)$$

where $D_{\text{sing}}(R) := D^b(R)/\operatorname{Perf}(R)$, due to Buchwetiz [Buc86], Eisenbud [Eis80], and Orlov [Orl04].

Since we also assume also that R has (at most) an isolated singularity, it follows that $\operatorname{Tor}_i^R(M, N)$ has finite length for $i \gg 0$. Since resulotions are eventually two-periodic, we have $\operatorname{Tor}_i^R(M, N) \cong \operatorname{Tor}_{i+2}^R(M, N)$ for $i \gg 0$ as well. We are lead to the defintion

$$\theta^R(M, N) := \operatorname{length} \operatorname{Tor}_{2i}^R(M, N) - \operatorname{length} \operatorname{Tor}_{2i+1}^R(M, N), i \gg 0$$

which was introduced by Hochster [Hoc81].

Just as $\chi(M, N)$ admits an interpretation in terms of intersection multiplicitiy, $\theta^R(M, N)$ has a nice geometric intepretation due to Buchweitz and van Staten [BVS12].

Theorem 1.3 (Buchweitz-van Staten). *If $R = \mathbb{C}[[x_0, \ldots, x_n]]/(f)$ is an isolated singulartiy, then for any pair of finitely generated R-modules M and N, we have*

$$\theta^R(M,N) = \operatorname{link}(ch(M), ch(N))$$

where $ch(M), ch(M)$ are certain classes in $H_n(L)$, the homology of the link L of the singularity, and link *is the classical linking form.*

By definition, the link is the intersection of the zero locus of f with a small sphere:

$$L := \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \mid f(z_0, \ldots, z_n) = 0\} \cap S^{2n+1}_\epsilon, \text{ for } 0 < \epsilon \ll 1,$$

where

$$S^{2n+1}_\epsilon := \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \mid \sum_i \|z_i\|^2 = \epsilon^2\}.$$

The linking form

$$\operatorname{link}(-,-) : H_n(L) \times H_n(L) \to \mathbb{Z}$$

measures how homology classes in L are "linked" inside the ambient sphere S^{2n+1}.

The following should be regarded as an analogue of the Serre Vanishing Conjecture:

Conjecture 1.4 (Dao-Kurano). *For an isolated hypersurface singularity R and finitely generated modules M and N, if $\dim(M) + \dim(N) \leq \dim(R)$ then $\theta(M,N) = 0$.*

This was proven by Dao [Dao13] in the "gemetric setting" – roughly, when R is of finite type over a field.

The focus of the second part of this talk will be a proof of this conjecture in full generality, using some techniques from K-theory. This proof is due to Brown, Miller, Thompson and myself [BMTW16a].

2. Basics of algebraic K-theory with supports

Throughout, Q is a commutative Noetherian ring and Z a closed subset of $\operatorname{Spec}(R)$. So, $Z = V(I)$ for some ideal I of Q.

Definition 2.1. Write $\operatorname{Perf}^Z(Q)$ for the category of bounded complexes of finitely generated and projective Q-modules whose homology is supported on Z. Morphisms are chain maps.

So, a typical object has the form

$$P := (\cdots \to 0 \to P_m \to \cdots \to P_n \to 0 \cdots)$$

for some $m, n \in \mathbb{Z}$, where each P_i finitely generated and projective over Q and the localized complex $P_\mathfrak{p}$ is acyclic for all $\mathfrak{p} \in \operatorname{Spec}(Q) \setminus Z$.

Definition 2.2. The *Grothendieck group of Q with supports on Z*, written $K_0^Z(Q)$, is the abelian group with the following presentation:

- The generators are isomorphism classes of objects of $\mathrm{Perf}^Z(Q)$.
- There are two types of relations:

$$[P] = [P']$$

if there is a homotopy equivalence $P \xrightarrow{\sim} P'$, and

$$[P] = [P'] + [P'']$$

if there is a short exaxt sequence of complexes $0 \to P' \to P \to P'' \to 0$.

Remark 2.3. A chain map $P \to P'$ of bounded complexes of projective modules is a quasi-isomorphism iff it is a homotopy equivalence.

Exercise 2.4. *Upon modding out the hom sets of* $\mathrm{Perf}^Z(Q)$ *by chain homotopy, one obtains a triangulated category* $\left[\mathrm{Perf}^Z(Q)\right]$. *Show* $K_0^Z(Q)$ *may equivalently be defined as the Grothendieck group of the triangulated category* $[\mathrm{Perf}^Z(Q)]$. *The latter is defined as the abelian group generated by isomorphism classes of objects modulo the relation* $[X] = [X'] + [X'']$ *for each distinguished triangle* $X' \to X \to X'' \to \Sigma X$.

Exercise 2.5. *Prove* $K_0^{\mathrm{Spec}\, Q}(Q) \cong K_0(Q)$, *via the map* $[P] \mapsto \sum_i (-1)^i [P_i]$, *where* $K_0(Q)$ *is the classical Grothendieck group of* Q.

Exercise 2.6. *Prove that if* Q *is regular local, then* $K_0^{\{\mathfrak{m}\}}(Q) \cong \mathbb{Z}$ *and is generated by the class of the Koszul complex on a regular sequence of genertors for* $\mathfrak{m}$.

Exercise 2.7. *For any commutative ring* Q, *element* $a \in Q$, *and positive integer* n, *let* $\mathrm{Kos}_Q(a) \in \mathrm{Perf}^{V(a)}(Q)$ *be the complex*

$$\cdots \to 0 \to Q \xrightarrow{a} Q \to 0 \to \cdots$$

concentrated in degrees 0 *and* 1. *Prove*

$$[\mathrm{Kos}_Q(a^n)] = n[\mathrm{Kos}_Q(a)].$$

2.1. **Basic properties: Functorality, pairings, Euler characteristic.**

2.1.1. *Functorality.* If $\phi : Q \to R$ is a ring map, let $\phi^* : \mathrm{Spec}(R) \to \mathrm{Spec}(Q)$ denote the induced map on spectra. Assume $Z \subseteq \mathrm{Spec}(Q)$ and $W \subseteq \mathrm{Spec}(R)$ are closed subsets such that $W \supseteq (\phi^*)^{-1}(Z)$. If $Z = V(I)$ and $W = V(J)$, this means that $J \subseteq \sqrt{I}R$. These hypotheses ensure that if P belongs to $\mathrm{Perf}^Z(Q)$ then $P \otimes_Q R$ belongs to $\mathrm{Perf}^W(R)$.

The two relations are preserved by this functor so that we have an induced map

$$\phi_* = \phi_*^{Z,W} : K_0^Z(Q) \to K_0^W(R).$$

We will use such functorality in just two special cases:

- If $\phi = \mathrm{id}_Q$ and $W \supseteq Z$, we get the map
$$K_0^Z(Q) \to K_0^W(Q)$$
induced by an inclusion of categories $\mathrm{Perf}^Z(Q) \subseteq \mathrm{Perf}^W(Q)$. The map on Grothendieck groups is *not* usually injective.
- If $R = Q[1/g]$, ϕ is the canonical map, and $W = Z \cap \mathrm{Spec}(Q[1/g])$, we get the map
$$K_0^Z(Q) \to K_0^{Z\setminus V(g)}(Q[1/g])$$
given by $[P] \mapsto [P[1/g]]$.

2.1.2. *Products.* Tensor product of complexes induces a bilinear map, which we refer to as "cup product", of the form

$$- \cup - : K_0^Z(Q) \times K_0^W(Q) \to K_0^{Z\cap W}(Q)$$

for each pair of closed subsets Z and W of $\mathrm{Spec}(Q)$. That is,

$$[P] \cup [P'] := [P \otimes_Q P'].$$

For example, if M and N are finitely generated Q-modules of finite projective dimension, then upon choosing bounded resolutsion P_M, P_N, we have

$$[P_M] \in K_0^{\mathrm{supp}(M)}(Q),\ [P_N] \in K_0^{\mathrm{supp}(N)}(Q)$$

and

$$[P_M] \cup [P_N] = [M \otimes_Q^{\mathbb{L}} N] \in K_0^{\mathrm{supp}(M)}(Q).$$

2.1.3. *Euler Characteristic.* For a maximal ideal $\mathfrak{m}$ of Q, if $P \in \mathrm{Perf}^{\mathfrak{m}}(Q)$, we set

$$\ell(P) = \sum_i (-1)^i \,\mathrm{length}\, H_i(P) \in \mathbb{Z}.$$

If $P \sim P'$ then clearly $\ell(P) = \ell(P')$ and the long exact sequence in homology associated to a short exact sequence of complexes shows that ℓ also preserves the other relation in the definition of $K_0^{\{\mathfrak{m}\}}(Q)$. We thus have an induced homomoprhism

$$\ell : K_0^{\mathfrak{m}}(Q) \to \mathbb{Z},\ \text{defined by } \ell([P]) = \ell(P).$$

If Q is regular local, this map is in fact an isomorphism by Exercise 2.6.

2.2. **K-theoretic interpretation of SVC.** There are perhaps many reasons one might be interested in the Grothendieck group with supports. In general, if an invariant of a module or of a pair of modules is actually an invariant of classes in the Grothendieck group, then one has hope that certain formal properties can be used to analyse the invaraint. This is true first and foremost of Serre's intersection multiplicity, as we now explain.

Given a local ring Q and a Q-module M of finite projetive dimension, we write $[M] \in K_0^{\operatorname{supp}(M)}(Q)$ for $[P_M]$, the class of a bounded projective resolution P_M of M over Q. (Any two such resolutions are homotopy equivalent and thus the class is well-defined.) For a pair of such modules M and N such that $\operatorname{supp}(M) \cap \operatorname{supp}(N) = \{\mathfrak{m}\}$ we have by defition

$$[M] \cup [N] = [P_M \otimes_Q P_N] \in K_0^{\mathfrak{m}}(Q).$$

Since $H_i(P_M \otimes_Q P_N) = \operatorname{Tor}_i^Q(M, N)$, the map

$$\ell : K_0^{\mathfrak{m}}(Q) \to \mathbb{Z}$$

sends $[M] \cup [N]$ to $\sum_j (-1)^j \operatorname{length} \operatorname{Tor}_j^Q(M, N)) = \chi(M, N)$. We have proven:

Proposition 2.8. *For a local ring Q and finitely generated Q-modules M and N each having finite projective dimension, if* $\operatorname{supp}(M) \cap \operatorname{supp}(N) = \{\mathfrak{m}\}$*, then*

$$\chi(M, N) = \ell([M] \cup [N]).$$

Thus, the statement of Serre's Vanishing Conjecture is "captured" by K-theory.

2.3. **Key right-exact sequence.**

Theorem 2.9 (Gillet-Soulé)**.** *For a regular ring Q, closed subset $Z \subseteq$* $\operatorname{Spec}(Q)$*, and element $g \in Q$, the canonical maps fit into a right-exact sequence*

$$K_0^{Z \cap V(g)}(Q) \to K_0^{Z}(Q) \to K_0^{Z \setminus V(g)}(Q[1/g]) \to 0.$$

Sketch of proof. For commutative Noetherian ring R, let $G_0(R)$ be the Grothendieck goup of finitely generated R-modules. This is the abelian group generated by isomorphism classes of such modules with relations coming from short exact sequences.

For any ideal I, there is an isomorphism

$$G_0(Q/I) \xrightarrow{\cong} K_0^{V(I)}(Q)$$

that sends $[M]$ to $[P_M]$ where P_M is a bounded projective resolution of M over Q. (Such a resolution exists since we assume Q is regular, and any two such resolutions are homotopy equivalent.)

It follows that the seqeunce in the statement is isomorphic to the sequence

$$G_0(R/g) \to G_0(R) \to G_0(R[1/g]) \to 0,$$

where $Z = V(I)$ and $R := Q/I$. Here, the left-hand map is induced by restriction of scalars along $R \twoheadrightarrow R/g$ and the right is induced by extension of scalars along $R \to R[1/g]$.) The latter is known to be right exact classically (see Bass). □

Remark 2.10. The sequence of the Theorem extends to the left using higher K-groups.

Remark 2.11. The assumption that Q is regular in the Theorem is essential, and its need is one of the key places that this assumption enters in to the proof of Serre's Vanishing Conjecture.

Corollary 2.12. *Given a regular ring Q, a closed subset $Z = V(I)$ of* $\mathrm{Spec}(Q)$*, and prime ideal $\mathfrak{p}$ that is minimal among those that contain I, there is a right-exact sequecne*

$$\bigoplus_{g \in Q \setminus \mathfrak{p}} K_0^{Z \cup V(g)}(Q) \to K_0^Z(Q) \to \mathbb{Z} \to 0.$$

where the map $K_0^Z(Q) \to \mathbb{Z}$ is the composition of

$$K_0^Z(Q) \xrightarrow{[P] \mapsto [P_\mathfrak{p}]} K_0^{\{\mathfrak{p}Q_\mathfrak{p}\}}(Q_\mathfrak{p}) \xrightarrow{\ell} \mathbb{Z}.$$

Proof. By the Theorem we have a right exact sequence

$$\bigoplus_g K_0^{Z \cup V(g)}(Q) \to K_0^Z(Q) \to \varinjlim_g K_0^{Z \setminus V(g)}(Q[1/g]) \to 0.$$

and since K-theory commutes with filtered colimits, there is an isomorphism

$$\varinjlim_g K_0^{Z \setminus V(g)}(Q[1/g]) \cong K_0^{\{\mathfrak{p}Q_\mathfrak{p}\}}(Q_\mathfrak{p}).$$

The result follow since $\ell : K_0^{\{\mathfrak{p}Q_\mathfrak{p}\}}(Q_\mathfrak{p}) \to \mathbb{Z}$ is an isomorphism by Exercise 2.6. □

3. The Gillet-Soulé Axioms

Definition 3.1. (Gillet-Soulé) Let $\mathcal{C}$ denote a full-subcategory of the category of commutative Noethering rings that is closed under localization; i.e., if $Q \in \mathcal{C}$ so is $S^{-1}Q$ for every multiplicatively closed subest S of Q. Given a positive integer k, an *Adams operation of degree k defined on $\mathcal{C}$* is a collection of functions

$$\psi^k_{Q,Z} : K_0^Z(Q) \to K_0^Z(Q)$$

for all $Q \in \operatorname{ob}\mathcal{C}$ and all closed subsets Z of $\operatorname{Spec}(Q)$ such that the following four axioms hold:

(1) (Compatibility with addition) $\psi^k_{Q,Z}$ is a homomorphism of abelian groups for all Q and Z.
(2) (Compatibility with multiplication) For any $Q \in \operatorname{ob}\mathcal{C}$ and pair of closed subsets Z and W, we have
$$\psi^k_{Q,Z\cap W}(\alpha \cup \beta) = \psi^k_{Q,Z}(\alpha) \cap \psi^k_{Q,W}(\beta)$$
for all $\alpha \in K_0^Z(Q)$ and $\beta \in K_0^W(Q)$.
(3) (Naturality) For any morphism $\phi : Q \to Q'$ in $\mathcal{C}$ and closed subset $Z \subseteq \operatorname{Spec}(Q)$ and $Z' \subseteq \operatorname{Spec}(Q')$ such that $(\phi^*)^{-1}(Z) \subseteq Z'$, we have
$$\psi^k_{Q',Z'}(\phi_*(\alpha)) = \phi_*\psi^k_{Q,Z}(\alpha) \text{ for all } \alpha \in K_0^Z(Q).$$
(4) (Normalization) For all a,
$$\psi^k_{Q,V(a)}([\operatorname{Kos}_Q(a)]) = k \cdot [\operatorname{Kos}_Q(a)] \text{ holds in } K_0^{V(a)}(Q),$$
where $\operatorname{Kos}_Q(a)$ is the complex $\cdots \to 0 \to Q \xrightarrow{a} Q \to 0 \to \cdots$ concentrated in degrees 0 and 1.

We will almost always write just ψ^k for any of the maps $\psi^k_{Q,Z}$.

Remark 3.2. Gilet-Soulé work in the category of all Noetherian schemes, but the modifications of the axioms are minor and left to the reader's imagination. (The fourth one involves only affine schemes.)

Example 3.3. Suppose ψ^k is a Adams operation of degree k. For any sequence $a_1, \ldots, a_c \in Q$ of elements, let $\operatorname{Kos}_Q(a_1, \ldots, a_c) = \bigotimes_i \operatorname{Kos}_Q(a_i)$ be the associated Koszul complex. We have
$$[\operatorname{Kos}_Q(a_1, \ldots, a_c)] \in K_0^{V(a_1,\ldots,a_c)}(Q).$$
By Axiom 2,
$$[\operatorname{Kos}_Q(a_1, \ldots, a_c)] = [\operatorname{Kos}_Q(a_1)] \cup \cdots \cup [\operatorname{Kos}_Q(a_c)]$$
with
$$[\operatorname{Kos}_Q(a_i)] \in K_0^{V(a_i)}(Q), \text{ for each } i.$$
Using also Axiom 4 we get
$$\psi^k([\operatorname{Kos}_Q(a_1, \ldots, a_c)]) = k^c[\operatorname{Kos}_Q(a_1, \ldots, a_c)].$$

In other words, $[\operatorname{Kos}_Q(a_1, \ldots, a_c)]$ is an eigenvector of eigenvalue k^c for any degree k Adams operation. One says that $[\operatorname{Kos}_Q(a_1, \ldots, a_c)])$ has *weight* c.

Theorem 3.4 (Gillet-Soulé)**.** *For each $k \geq 1$, there exists an Adams operation ψ^k_{GS} of degree k defined on the category of all commutative Noetherian rings. Moroever, these operators satisfy*

$$\psi^k_{GS} \circ \psi^j_{GS} = \psi^{jk}_{GS}.$$

for all $j, k \geq 1$. In particular, they commute.

We won't prove this theorem, but do offer a few comments on their construction. Gillet-Soulé first establish "λ operations", which are induced from the exterior power functors. These are not defined by taking exterior powers of complexes themselves, but rather by taking exterior powers of simplicial Q-modules, using the Dold-Kan correspondence. The idea behind this goes back to Dold and Puppe. Having defined such operations, they verify via a difficult argument that the axioms of a (special) λ ring are satisfied. The operation ψ^k is then defined by a universal formula from the λ-operations.

Exercise 3.5. *Fix a prime p and let $\mathcal{C}$ be the category of commutative Noetherian rings having charactertistic p. For each $Q \in \mathcal{C}$, let $F : Q \to Q$ denote the Frobenious endomorphism. Since $F^* : \operatorname{Spec}(Q) \to \operatorname{Spec}(Q)$ is a homeomorhism, for each closed subset Z we have an induced map*

$$F_* : K_0^Z(Q) \to K_0^Z(Q)$$

that sends $[P]$ to $[P \otimes_Q {}^F Q]$. Prove F_* is an Adams operation of weight p defined on $\mathcal{C}$.

Example 3.6. Recall that if $(Q, \mathfrak{m})$ is regular local, then $K_0^{\mathfrak{m}}(Q) \cong \mathbb{Z}$, and is generated by the class of the Koszul complex. Thus any Adams operation of degree k acts as k^d on $K_0^{\mathfrak{m}}(Q)$ where $d = \dim(Q)$.

Theorem 3.7 (Gillet-Soulé)**.** *Let $\mathcal{C}$ be a category of rings as above and ψ^k an Adams operation of degree k defined on $\mathcal{C}$, where $k \geq 2$. For any regular ring Q belonging to $\mathcal{C}$ and any closed subset Z of $\operatorname{Spec}(Q)$, we have*

$$K_0^Z(Q) \otimes_{\mathbb{Z}} \mathbb{Q} = \bigoplus_{j=\dim(Q)-\dim(Z)}^{\dim(Q)} K_0^Z(Q)^{(j)}$$

where

$$K_0^Z(Q)^{(j)} := \ker(K_0^Z(Q) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\psi^k - k^j} K_0^Z(Q) \otimes_{\mathbb{Z}} \mathbb{Q}).$$

In other words, the action of ψ^k on $K_0^Z(Q) \otimes_{\mathbb{Z}} \mathbb{Q}$ is diagonalizable with eigenvalues belonging to the set

$$\{k^j \mid \dim(Q) - \dim(Z) \leq j \leq \dim(Q)\}.$$

Elements of $K_0^Z(Q)^{(j)}$ are said to have *weight* j (for the operator ψ^k).

Sketch of Proof. Given regular ring in Q in $\mathcal{C}$, consider those ideals I of Q such that the action of ψ^k on $K_0^{V(I)}(Z)$ fails to be diagonalizable with all eigenvalues belonging to $\{k^j \mid \mathrm{ht}(I) \leq j \leq \dim(Q)\}$. By the Noetherian property, if this is a non-empty collection there is a maximal member I. Set $c = \mathrm{ht}(I)$. Let $\mathfrak{p}$ be minimal among those primes containing I. By Corollary 2.12 and the naturality of the operator ψ^k, we have the right exact sequence

$$\bigoplus_{g \in Q \setminus \mathfrak{p}} K_0^{V(I,g)}(Q) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0^{V(I)}(Q) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}(c) \to 0.$$

of $\mathbb{Q}[\psi^k]$-modules. Here $\mathbb{Q}(c) = \mathbb{Q}$ with ψ^k acting as k^c. Since $(I,g) \supset I$, by the choice of I we have that ψ^k acts diagonally on

$$\bigoplus_{g \in Q \setminus \mathfrak{p}} K_0^{V(I,g)}(Q) \otimes_{\mathbb{Z}} \mathbb{Q}$$

with eigenvalues bellonging to the set

$$S := \{k^j \mid c < j \leq \dim(Q)\}.$$

It follows that ψ^k also acts diagaonally with these eigenvalues on the image of $\bigoplus_{g \in Q \setminus \mathfrak{p}} K_0^{V(I,g)}(Q) \otimes_{\mathbb{Z}} \mathbb{Q}$ in $K_0^{V(I)}(Q) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Since $k^c \notin S$, the result now follows from basic linear algebra: Given a short exact sequence of vector spaces

$$0 \to V' \to V \to V'' \to 0$$

and compatible endomorphisms ψ', ψ and ψ'', if ψ' and ψ'' are diagaonlizable and have no eigenvalues in common, then ψ is also diagonalizable and its eigenvalues are the union of those of ψ' and ψ''. $\square$

Exercise 3.8. *Suppose Q is a regular ring in $\mathcal{C}$, ψ^k is a degree k Adams operation for $k \geq 2$, and ϕ^j is a degree j Adams operation for $j \geq 2$. (The possibility that $j = k$ is allowed.) Prove that if ψ^k and ϕ^j commute, then*

$$K_0^Z(Q)^{(j)_\psi} \otimes_{\mathbb{Z}} \mathbb{Q} = K_0^Z(Q)^{(j)_\phi} \otimes_{\mathbb{Z}} \mathbb{Q}$$

where the left-hand side refers to the weight space coming from ψ^k and the right-hand side refers to the weight space coming from ϕ^j.

In particular if $j = k$ and ψ^k and ϕ^j commute, then $\psi^k \otimes_{\mathbb{Z}} \mathbb{Q} = \phi^j \otimes_{\mathbb{Z}} \mathbb{Q}$.

Remark 3.9. I do not know if the assumption that ψ^k and ϕ^j commute is necessary.

Exercise 3.10. *Prove that if p is a prime and $\mathcal{C}$ is the category of commutative Noetherian rings of charactistic p, then for any Adams operation ψ^p of degree p defined on $\mathcal{C}$, we have $\psi^p \otimes_{\mathbb{Z}} \mathbb{Q} = F \otimes_{\mathbb{Z}} \mathbb{Q}$ where F is the degree p Adams operation induced by Frobenius.*

4. Cyclic Adams operations

In this section I present a construction a degree p Adams operation on $K_0^Z(Q)$ recently developed by my co-authors and me. We call these "cyclic Adams operations", and write them as ψ^p_{cyc}. Gillet-Soulé long ago proved such operators exist in complete generality, and ours exist on only a slighlty more restrictive setting. So, one might ask "What's the point"'? There are two points:

- The construction of the cyclic Adams operations is simpler and the verification that they satisfy the four axioms is easier.
- The construction does not rely on the Dold-Kan correspondence and thus it can be ported to other settings in which such a corredondence does not exist.

The latter point will be illustrated when we discuss cyclic Adams operations on matrix factorizations.

For a fixed prime p, set $A_p = \mathbb{Z}[\frac{1}{p}, \zeta_p]$ where $\zeta_p = e^{\frac{2\pi i}{p}}$, a primitive p-th root of unity. We work in the category $\mathcal{C}$ consisting of commutative, Noetherian A_p-algebras. For $Q \in \mathcal{C}$, given a bounded complex of projectives Q-modules P, let

$$T^p(P) := \overbrace{P \otimes_Q \cdots \otimes_Q P}^{p}$$

and endow $T^p(P)$ with the following action of $C_p = \langle \sigma \rangle$, the cyclic group of order p. We think of σ as being the p-cycle $(1\,2\,\ldots,p)$ and let it act on an element of $T^p(P)$ by

$$\sigma \cdot (x_1 \otimes \cdots \otimes x_p) = (-1)^{|x_1|(|x_2|+\cdots+|x_p|)} x_2 \otimes \cdots \otimes x_p \otimes x_1$$

Since $1/p, \zeta_p \in Q$, we have

$$T^p(P) = \bigoplus_{j=0}^{p-1} T^p(P)^{(\zeta_p^j)}$$

where

$$T^p(P)^{(w)} := \ker(T^p(P) \xrightarrow{\sigma - w} T^p(P)),$$

the eigenspace of eigenvalue w for the action of σ. It is not hard to see that if P is supported on Z, so is $T^p(P)^{(w)}$ for each w. Thus, for

$P \in \mathrm{Perf}^Z(Q)$, we can define

$$\psi^p_{\mathrm{cyc}}(P) = [T^p(P)^{(1)}] - [T^p(P)^{(\zeta_p)}] \in K_0^Z(Q).$$

Remark 4.1. Equivalently,

$$\psi^p_{\mathrm{cyc}}(P) = \sum_{j=0}^{p-1} \zeta_p^j [T^p(P)^{(\zeta_p^j)}]$$

which looks slightly more natural.

Theorem 4.2. *For any commutative, Noetherian A_p-algebra Q and closed subset Z of $\mathrm{Spec}(Q)$, ψ^p_{cyc} determines a function*

$$\psi^p_{\mathrm{cyc}} : K_0^Z(Q) \to K_0^Z(Q),$$

given by $\psi^p_{\mathrm{cyc}}([P]) = [T^p(P)^{(1)}] - [T^p(P)^{(w)}]$.

Moreover ψ^p_{cyc} satsifies the Gillet-Soulé axioms on the category $\mathcal{C}$ of all commutative Noetherian A_p-algebras.

5. The Proof of SVC in the regular case

Theorem 5.1 (Serre's Vanishing Conjecture for Regular Rings)**.** *Suppose Q is a regular local ring, M and N are finitely generaty Q-modules such that $M \otimes_Q N$ has finite length. If $\dim(M) + \dim(N) < \dim(Q)$ then*

$$\chi(M, N) = 0.$$

Gillet-Soule's Proof, using cyclic Adams operations. Choose a prime $p \notin \mathfrak{m}$ so that $1/p \in Q$. There is a finite map $Q \to Q'$ of local rings such that $\zeta_p \in Q'$ and we have

$$\chi^Q(M, N) = \lambda \cdot \chi^{Q'}(M \otimes_Q Q', N \otimes_Q Q').$$

for a non-zero number λ. We may therefore assume without loss of generality that $\zeta_p \in Q$, so that ψ^p_{cyc} is defined.

We have

$$[M] \in K_0^{\mathrm{supp}(M)}(Q) \otimes_{\mathbb{Z}} \mathbb{Q} = \bigoplus_{j=c_M}^{d} K_0^{\mathrm{supp}(M)}(Q)^{(j)}$$

and

$$[N] \in K_0^{\mathrm{supp}(N)}(Q) \otimes_{\mathbb{Z}} \mathbb{Q} = \bigoplus_{j=c_N}^{d} K_0^{\mathrm{supp}(N)}(Q)^{(j)}$$

where $d = \dim(Q)$, $c_M = \dim(Q) - \dim(\operatorname{supp}(M))$, and $c_N = \dim(Q) - \dim(\operatorname{supp}(N))$. By Axioms 2 and 3,

$$[M] \cup [N] \in \bigoplus_{i \geq c_M, j \geq c_N} K_0^{\mathfrak{m}}(Q)^{(i+j)}.$$

Since $\dim(M) + \dim(N) < d$, we have $c_M + c_N > d$. But $K_0^{\mathfrak{m}}(Q) \otimes_{\mathbb{Z}} \mathbb{Q} = K_0^{\mathfrak{m}}(Q)^{(d)}$, and thus $[M] \cup [N] = 0$ in $K_0^{\mathfrak{m}}(Q)$. The theorem thus follows from Proposition 2.8. □

6. Matrix Factorizations and the Theta invariant

6.1. **The theta invariant.** We assume throughout that R is an *isolated hypersurface singularity (i.h.s.) ring* by which we mean $R = Q/f$ for a regular local ring Q and some $f \in Q$ and that $R_{\mathfrak{p}}$ is regular for all $\mathfrak{p} \neq \mathfrak{m}$. We recall from the introduction that, since R is a hypersurface

> the resulotuion of any finitely generated R-module is eventually two-periodic

and since R is an isolated singularity

> $\operatorname{Tor}_i^R(M, N)$ is finite length for $i \gg 0$ for any pair of finitely generated R-modules.

The properties listed above lead to:

Definition 6.1 (Hochster)**.** For an i.h.s. ring R, given two finitely generated R-modules M and N, define

$$\theta(M, N) = \text{length} \operatorname{Tor}_{2i}^R(M, N) - \text{length} \operatorname{Tor}_{2i+1}^R(M, N), \text{ for } i \gg 0.$$

Conjecture 6.2 (Dao-Kurano)**.** *If* $\dim(M) + \dim(N) \leq \dim(R)$ *then* $\theta(M, N) = 0$.

6.2. **Matrix factorizations.** As before $R = Q/f$ with Q regular local. The eventual two periodicity of resolutions over F is recorded by *matrix facotozations*. We defined a category $mf(Q, f)$ that is analogous to $\operatorname{Perf}(Q)$ with two differences:

- we work with $\mathbb{Z}/2$-graded objects instead of $\mathbb{Z}$-graded ones,
- the difference squares to f instead of 0.

Here is the formal definition:

Definition 6.3. For any commutative ring Q and element $f \in Q$, $mf(Q, f)$ is the category whose objects are a pair of f.g. projective Q-modules and maps between them,

$$P_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} P_0$$

such that $d_0 \circ d_1 = f\,\mathrm{id}_{P_1}$ and $d_1 \circ d_0 = f\,\mathrm{id}_{P_0}$. Morphisms are the analogues of chain maps: i.e., they are given by a pair of maps causing both triangles to commute.

Of course, if Q is local, then each of P_0, P_1 is free of finite rank and, upon choosing basis, we may represent an object of $mf(Q, f)$ as a pair of $n \times n$ matrices A, B such that $AB = fI_n = BA$

There is also a notion of homotopy of morphisms, defined analogously to chain homotopes. The resulting homotopy category is written $[mf(Q, f)]$.

Given a matrix fatorization $P_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} P_0$ we get an associated R-module (where $R = Q/f$) by taking $\mathrm{coker}(P_1 \xrightarrow{d_1} P_0)$, and in fact this R-module is MCM.

Theorem 6.4 (Buchweitz-Eisenbud-Orvlov)**.** *If Q is regular and $f \in Q$ is a non-zero-divisor, there is an equivalence of triangulated categories*

$$[mf(Q, f)] \cong D^b(R)/\operatorname{Perf}(R) \cong \underline{MCM}(R)$$

where $R = Q/f$.

6.3. **Proof of the conjecture of Dao-Kurano.**

Theorem 6.5 (Brown-Miller-Thompson-W)**.** *Suppose $R = Q/f$ for a regular local ring Q and non-zero element $f \in Q$. If M and N are finitely generated R-modules such that* $\dim(M) + \dim(N) \leq \dim(R)$, *then $\theta(M, N) = 0$.*

I give an outline of the proof:

- Using the Buchweitz-Eisenbud-Orlov Theorem, we may realize the θ invariant as a pairing of the form
$$K_0(mf(Q, f)) \times K_0(mf(Q, -f)) \xrightarrow{-\cup-} K_0^{\mathfrak{m}}(mf(Q, 0)) \xrightarrow{\ell} \mathbb{Z}$$
where $K_0^{\mathfrak{m}}(mf(Q, 0))$ is the Grothendieck group of $\mathbb{Z}/2$-graded complexes of Q-modules supported on $\{\mathfrak{m}\}$. That is,
$$\theta(M, N) = \ell([M]_{stable} \cup [N]_{stable})$$
where $[M]_{stable}$ is the class of a matrix factorization representing M in the stable category, and similarly for $[N]_{stable}$. (As a technical point, we realize the latter class in $K_0(Q, -f)$ using that $Q/(-f) = Q/f = R$.)
- In parallel with the proof of SVC, we may assume there is a prime p such that $1/p, \zeta_p \in Q$. In this situation we define a cyclic Adams operator
$$\psi^p_{\mathrm{cyc}} : K_0(mf(Q, f)) \to K_0(mf(Q, f)).$$

The definition is very close to the one described on $K_0^Z(Q)$ above. Namely, for a matrix factorization $P = (P_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} P_0$, we define

$$\psi^p_{\text{cyc}}(P) = [T^p(P)^{(1)}] - [T^p(P)^{(\zeta_p)}].$$

(This class actually belontg to $K_0(mf(Q, p \cdot f))$ but there is an evident isomorphism $K_0(mf(Q, p \cdot f)) \cong K_0(mf(Q, f))$.)

- These cyclic Adams operations satisfy axioms analogous to the four Gillet-Soulé axioms, and moreover the action of $\psi^p_{\text{cyc}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is diagonalizable with appropiated eigenvalues. This leads to a proof that if $\dim(M) + \dim(N) \leq \dim(R)$ then

$$[M]_{stable} \cup [N]_{stable} \in \bigoplus_{j > \dim(Q)} K_0^{\mathfrak{m}}(mf(Q, 0))^{(j)}$$

- We do *not* know whether

$$K_0^{\mathfrak{m}}(mf(Q, 0)) \otimes_{\mathbb{Z}} \mathbb{Q} = K_0^{\mathfrak{m}}(mf(Q, 0))^{(d)}.$$

  If so, the proof would be complete. But we do know that the map

$$\ell : K_0^{\mathfrak{m}}(mf(Q, 0)) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$$

  satisfies $\ell \circ \psi^p_{\text{cyc}} = k^d \cdot \ell$, which suffices to complete the proof.

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