UNIFORMITY IN
REDUCTION TO CHARACTERISTIC P

FELIPE PÉREZ, KEVIN TUCKER, AND YONGWEI YAO

Abstract. This note is a preliminary version of a section of an article in preparation. Herein, we show certain uniform upper bounds for Hilbert-Kunz Multiplicity: see Theorem 2.11 and Theorem 2.13. Updated April 26, 2021.

2. Uniform Upper Bounds for Hilbert-Kunz Multiplcity

2.1. **Hilbert-Kunz Numbers and Multiplicity.** Throughout this article, all rings will be assumed Noetherian and all schemes Noetherian and separated. For a ring R with prime characteristic $p > 0$, we let $F\colon R \to R$ denote the Frobenius endomorphism or p^{th} power mapping. If $I \subseteq R$ is an ideal and $q = p^e$ with $e \geqslant 0$, then $I^{[q]} = \langle F^e(I)\rangle$ denotes the expansion of I along the e-iterated Frobenius. For an R-module M and $e \geqslant 0$, we write F^e_*M for the R-module induced by restriction of scalars for the e-iterated Frobenius $F^e : R \to R$. Thus, F^e_*M agrees with M as an abelian group, and we have $r \cdot F^e_*m = F^e_*(r^{p^e} \cdot m)$ for $r \in R$ and $m \in M$. Note that $F^e_*(\_\_)$ is an exact functor on the category of R-modules. When R is reduced, F^e_*R is naturally identified with R^{1/p^e} as a ring, so that F^e takes on the guise of the inclusion $R \subseteq R^{1/p^e}$. We say that R is F-finite if the Frobenius is a finite morphism, or equivalently F^e_*R is a finitely generated R-module for all $e \geqslant 0$. An F-finite ring is excellent [Kun76] and admits a dualizing complex [Gab04].

We shall also make use of another commonly used notational convention, denoting by ${}^eM$ the R-bimodule whose left structure is given by F^e and whose right structure is the usual one. Thus, in ${}^eM$, we have $r \cdot m = m \cdot r^{p^e}$. We use $\ell(\_\_)$ to denote the length of an R-module, as well as $\ell^l(\_\_)$ and $\ell^r(\_\_)$ for the left and right lengths of a bimodule as needed. Similarly, we use $\mu(\_\_)$ to denote the minimal number of generators of an R-module, with $\mu^l(\_\_)$ and $\mu^r(\_\_)$ denoting the respective left and right numbers for a bimodule as necessary.

By a local ring $(R, \mathfrak{m})$, we mean a Noetherian ring R with a unique maximal ideal $\mathfrak{m}$; let κ denote the residue field $R/\mathfrak{m}$. When R has prime characteristic $p > 0$ and is F-finite, we have

$$\ell^l({}^eM) = \ell_R(F^e_*M) = [\kappa : \kappa^{p^e}] \cdot \ell_{F^e_*R}(F^e_*M) = [\kappa : \kappa^{p^e}] \cdot \ell^r({}^eM)$$

for all $e \geqslant 1$. Similarly, we have

$$\mu^l({}^eM) = [\kappa : \kappa^{p^e}] \cdot \ell(M/\mathfrak{m}^{[p^e]}) = [\kappa : \kappa^{p^e}] \cdot \mu^r({}^eM)$$

Recall also that for an R-module M *the dimension of* M is defined as $\dim(M) := \dim(R/\operatorname{Ann}_R(M))$.

Definition 2.1. If M is a finitely generated module over a local ring $(R, \mathfrak{m})$ of prime characteristic $p > 0$ and $e \geqslant 1$, let

$$f_e(M) = \frac{1}{p^{e \cdot \dim(M)}} \ell(M/\mathfrak{m}^{[p^e]}M) = \frac{1}{p^{e \cdot \dim(M)}} \ell^r\left((R/\mathfrak{m}) \otimes_R {}^e M\right)$$

denote the *e-th normalized Hilbert-Kunz number* of M.

Kunz [Kun69] initated the study of the colengths of the Frobenius powers of the maximal ideal. It is a theorem of Monsky [Mon83] that the limit of the normalized Hilbert-Kunz numbers exists; this limit is called the *Hilbert-Kunz multiplicity* of M and it is denoted by $\mathrm{e}_{\mathrm{HK}}(M) := \lim_{e\to\infty} f_e(M)$. Among the most important consequences of these definitions is a characterization of regularity.

Theorem 2.2 ([Kun69, Theorem 2.1, Proposition 3.2, Theroem 3.3], [WY04, Theorem 1.5], also cf. [HY02])**.** *If $(R, \mathfrak{m}, k)$ be a Noetherian local ring of prime characteristic p, then*

(a) $f_e(R) \geqslant 1$ *for all* $e \geqslant 1$.
(b) *R is a regular local ring if and only if $f_e(R) = 1$ for one (or equivalently for all) $e \geqslant 1$.*
(c) *R is regular local ring if and only if* $\mathrm{e}_{\mathrm{HK}}(R) = 1$ *and R is unmixed.*

The (normalized) Hilbert-Kunz numbers and the Hilbert-Kunz multiplicity can be extended to functions on the spectrum of any Noetherian ring R of prime characteristic p; for every $P \in \operatorname{Spec}(R)$ and $q = p^e$, we accordingly have

$$f_e(M_P) = \ell_{R_P}(M_P/P^{[q]}M_P)/q^{\dim(M_P)} \quad \text{and} \quad \mathrm{e}_{\mathrm{HK}}(M_P) = \lim_{e\to\infty} f_e(M_P),$$

the e^{th} normalized Hilbert-Kunz number of M_P and the Hilbert-Kunz multiplicity of M_P, respectively. When additionally R is an F-finite domain R with fraction field K, the Hilbert-Kunz numbers also have an important alternate interpretation. It was shown by Kunz in [Kun76] that $[K : K^p] = [\kappa(P) : \kappa(P)^p] \cdot p^{\mathrm{ht}(P)}$ for all $P \in \operatorname{Spec}(R)$ where $\kappa(P) = R_P/PR_P$ is the residue field at P. It follows that the Hilbert-Kunz numbers equal the ratio

$$f_e(R_P) = \frac{\min\{u \geqslant 0 \mid \text{there exists a surjection } R_P^{\oplus u} \twoheadrightarrow R_P^{1/q}\}}{\operatorname{rank}_{R_P}(R_P^{1/q})}$$

between the minimal number of generators of $R_P^{1/q}$ and the torsion-free rank of $R_P^{1/q}$.

Remark 2.3. More generally, for an F-finite ring R of prime characteristic $p > 0$, denote by $\alpha(P) := \log_p([\kappa(P) : \kappa(P)^p])$ for $P \in \operatorname{Spec}(R)$ where $\kappa(P) = R_P/PR_P$ is the residue field at P. When R is locally equidimensional, it was shown by Kunz in [Kun76] that the quantity $\alpha(P) + \mathrm{ht}(P)$ is constant on each of the connected components of $\operatorname{Spec}(R)$ (cf. [EY11, Page 4], [SB79]).

2.2. **Uniform Upper Bounds.** Our primary goal in this section is to exhibit certain uniform upper bounds for the Hilbert-Kunz numbers in various settings. To that end, we will repeatedly need to apply and leverage the Noether Normalization Lemma.

Theorem 2.4 (Noether Normalization, cf. [Eis95, Corollary 16.18]). *Let A be an integral domain and let R be any finitely generated A-algebra extension of A. Then there is a nonzero element $a \in A$ and elements $z_1, \dots, z_d$ in R algebraically independent over $A[a^{-1}]$ such that $R[a^{-1}]$ is module-finite over its subring $A[a^{-1}][z_1, \dots, z_d]$. Furthermore, if R is a domain generically separable over A, then there is $a \in A$ such that $R[a^{-1}]$ is separable over $A[a^{-1}][z_1, \dots, z_d]$.*

Lemma 2.5. *Let R be a Noetherian ring and S is a module finite R-algebra.*

(a) *Suppose I is an ideal of R with $\ell_R(R/I) < \infty$ and J an ideal of S with $IS \subseteq J$. For all $Q \in \operatorname{Spec}(S)$, we have*

$$\ell_{S_Q}(S_Q/JS_Q) \leqslant \ell_S(S/J) \leqslant \ell_S(S/IS) \leqslant \ell_R(S/IS) \leqslant \mu_R(S) \cdot \ell_R(R/I).$$

(b) *In prime characteristic p, if $Q \in \operatorname{Spec}(S)$ with $Q \cap R = P$, for all $e \geqslant 1$ we have*

$$\ell_{S_Q}(S_Q/Q^{[p^e]}S_Q) \leqslant \mu_R(S) \cdot \ell_{R_P}(R_P/P^{[p^e]}R_P).$$

If moreover $\operatorname{ht}(P) = \operatorname{ht}(Q)$, then $f_e(S_Q) \leqslant \mu_R(S) \cdot f_e(R_P)$.

Proof. The inequalities in (a) are straightforward, with the last in the string coming from tensoring a given surjection $R^{\oplus \mu_R(S)} \twoheadrightarrow S$ with R/I; (b) is an immediate application of (a). □

Lemma 2.6. *If $A \subseteq R$ is a module finite extension of Noetherian rings where A is regular domain and R is torsion free over A, then for every $Q \in \operatorname{Spec}(R)$ we have*

$$f_e(R_Q) \leqslant \mu_A(R).$$

Proof. Letting $P = Q \cap A$, we have have that $\operatorname{ht}(P) = \operatorname{ht}(Q)$ as $A \subseteq R$ has the going down property. Applying Lemma 2.5 gives $f_e(R_Q) \leqslant \mu_A(R) \cdot f_e(A_P)$ and the conclusion follows as $f_e(A_P) = 1$ for all $P \in \operatorname{Spec}(A)$. □

Lemma 2.7. *Let R be a Noetherian ring and M a finitely generated R-module with a filtration*

$$0 = M_n \subseteq M_{n-1} \subseteq \cdots \subseteq M_i \subseteq \cdots \subseteq M_1 \subseteq M_0 = M.$$

Denote $N_i := M_{i-1}/M_i$ for $i = 1, \dots, n$.

(a) *If T is an R-algebra such that $\ell(N_i \otimes_R T) < \infty$ for all $i = 1, \dots, n$, then $\ell(M \otimes_R T) < \infty$ and*

$$\ell(M \otimes_R T) \leqslant \sum_{i=1}^{n} \ell(N_i \otimes_R T)$$

(b) *In particular, if R has prime characteristic $p > 0$, then for all $P \in \operatorname{Spec}(R)$ and all $e \geqslant 1$ we have*

$$f_e(M_P) \leqslant \sum_{i=1}^{n} f_e((N_i)_P).$$

Proof. For every right exact sequence of R-modules

$$E' \to E \to E'' \to 0$$

and R-algebras T so that $E' \otimes_R T$ and $E'' \otimes_R T$ are finite length, tensoring by T gives a right exact sequence

$$E' \otimes_R T \to E \otimes_R T \to E'' \otimes_R T \to 0$$

yielding

$$\ell(E'' \otimes_R T) \leqslant \ell(E \otimes_R T) \leqslant \ell(E' \otimes_R T) + \ell(E'' \otimes_R T)$$

and (1) follows by induction on the length n of the filtration of M. For $P \in \operatorname{Spec}(R)$ and $e \geqslant 1$, applying (a) with $T = R_P/P^{[p^e]}R_P$ gives

$$\ell(M_P/P^{[p^e]}M_P) \leqslant \sum_{i=1}^{n} \ell((N_i)_P/P^{[p^e]}(N_i)_P).$$

Since $M_\bullet \otimes_R R_P$ remains a filtration of M_P, it follows $\dim(M_P) = \max_i \dim((N_i)_P)$ and hence

$$f_e(M_P) \leqslant \sum_{i=1}^{n} f_e((N_i)_P)$$

as $p^{e\cdot\dim(M_P)} \geqslant p^{e\cdot\dim((N_i)_P)}$ for all $i = 1, \ldots, n$. □

Proposition 2.8. *Suppose that k is a field of characteristic $p > 0$, R is a finitely generated k-algebra, and M is a finitely generated R-module. There exists a constant C such that for all $P \in \operatorname{Spec}(R)$ and all $e \geqslant 1$ we have $f_e(M_P) \leqslant C$, that is*

$$\ell(M_P/P^{[p^e]}M_P) \leqslant C \cdot p^{e\cdot\dim(M_P)}.$$

Proof. Consider a prime cyclic filtration

$$0 = M_n \subseteq M_{n-1} \subseteq \cdots \subseteq M_i \subseteq \cdots \subseteq M_1 \subseteq M_0 = M.$$

of M, with $Q_i \in \operatorname{Spec}(R)$ satisfying $M_{i-1}/M_i \simeq R/Q_i$ for $i = 1, \ldots, n$. As each R/Q_i remains a finitely generated k-algebra, we can exhibit them as module finite extensions $A_i \subseteq R/Q_i$ where A_i is polynomial over k. Applying Lemma 2.7 and Lemma 2.6, we have

$$f_e(M_P) \leqslant \sum_{i=1}^{n} f_e((R/Q_i)_P) \leqslant \sum_{i=1}^{n} \mu_{A_i}(R/Q_i)$$

yielding the constant $C := \sum_{i=1}^{n} \mu_{A_i}(R/Q_i)$ that is independent of $P \in \operatorname{Spec}(R)$ and $e \geqslant 1$. □

Our next goal is to show that in fact Proposition 2.8 holds more generally for excellent rings, where we can leverage that the regular locus is open.

Theorem 2.9 (Open regular locus [Mat80, Chapter 13])**.** *The following holds:*

- *If R is an excellent domain (or reduced), then there exits a nonzero divisor $x \in R$ such that $R_x := R[x^{-1}]$ is regular.*

- *If R is finitely generated A-algebra that is a domain of characteristic zero, then there exists a nonzero divisor $x \in R$ such that $R_x := R[x^{-1}]$ is smooth over A, which implies that the image of x in $R_{\mathfrak{p}} := R \otimes_A \kappa(\mathfrak{p})$ is in the defining ideal of the non-smooth locus of $R_{\mathfrak{p}}$ over $\kappa(\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec}(A)$.*

Lemma 2.10. *Let $(R, \mathfrak{m})$ be a local ring of prime characteristic p and $x \in \mathfrak{m}$ such that $\dim(R/(x)) = \dim(R) - 1$. Then $f_e(R) \leqslant f_e(R/(x))$ for all $e \geqslant 1$.*

Proof. For each $e \geqslant 1$, the successive quotients of the length p filtration

$$\mathfrak{m}^{[p^e]} = \mathfrak{m}^{[p^e]} + (x^{p^e}) \subseteq \mathfrak{m}^{[p^e]} + (x^{p^e-1}) \subseteq \cdots \subseteq \mathfrak{m}^{[p^e]} + (x^i) \subseteq \cdots \subseteq \mathfrak{m}^{[p^e]} + (x) \subseteq R$$

are themselves quotients of $R/(x, \mathfrak{m}^{[p^e]})$, so we see

$$\ell(R/\mathfrak{m}^{[p^e]}) \leqslant p \cdot \ell(R/(x, \mathfrak{m}^{[p^e]}))$$

and the result follows by dividing through by $p^{e \cdot \dim(R)}$. □

Theorem 2.11. *For every finitely generated module M over any excellent ring R of prime characteristic p, there exists a constant C such that, for all $P \in \operatorname{Spec}(R)$, and all $e \geqslant 1$,*

$$f_e(M_P) \leqslant C \qquad \text{or equivalently} \qquad \ell_{R_P}(M_P/P_P^{[p^e]}M_P) \leqslant C \cdot (p^e)^{\dim(M_P)}.$$

Proof. Suppose that there exists a counterexample, say obtained with M over R. By Noetherian induction (on R), we may assume that the theorem holds for all proper quotient rings $\bar{R}$ of R and all finitely generated modules $\bar{M}$ over $\bar{R}$.

Consider a prime cyclic filtration

$$0 = M_n \subseteq M_{n-1} \subseteq \cdots \subseteq M_i \subseteq \cdots \subseteq M_1 \subseteq M_0 = M.$$

of M, with $Q_i \in \operatorname{Spec}(R)$ satisfying $M_{i-1}/M_i \simeq R/Q_i$ for $i = 1, \ldots, n$. Using Lemma 2.7, we see that the R-module R/Q_i must also be a counterexample for some i. By assumption, this is only possible if $Q_i = 0$, so we see that R is a domain and constitutes a counterexample as a module over itself.

As R is an excellent domain, the regular locus is open and nonempty, so we may take an element $0 \neq x \in R$ with $R[x^{-1}]$ regular. Since $R/(x)$ is a proper quotient of R, there is a constant $C \geqslant 1$ so that

$$f_e((R/(x))_P) \leqslant C$$

for all $P \in \operatorname{Spec}(R/(x))$ and all $e \geqslant 1$. Since R is a counterexample as a module over itself, for some $P \in \operatorname{Spec}(R)$ and $e \geqslant 1$ we have $f_e(R_P) > C$. If $x \notin P$, then R_P is regular and we have a contradiction as $f_e(R_P) = 1 \leqslant C$. Thus we must have $x \in P$, but this leads to a contradiction as well since

$$f_e(R_P) \leqslant f_e((R/(x))_P) \leqslant C$$

by Lemma 2.10. □

Finally, to conclude this section and combining together all of the techniques used in the above proofs, we wish to show that the uniform upper bounds on the Hilbert-Kunz numbers are also valid in families.

Theorem 2.12 (Grothendieck's Generic Freeness Lemma, [Eis95, Theorem 14.4])**.** *Suppose that R is a Noetherian domain and S is a finitely generated R-algebra. If M is a finitely generated S-module, then there exists an element $a \neq 0$ in R such that $M[a^{-1}]$ is a free $R[a^{-1}]$-module.*

Theorem 2.13. *For every Noetherian ring A, every finitely generated A-algebra R, and every finitely generated R-module M, there exists a constant C that satisfies the following: for all $\mathfrak{p} \in \operatorname{Spec}(A)$ whose residue field $\kappa(\mathfrak{p})$ has positive characteristic, all regular $\kappa(\mathfrak{p})$-algebras Λ, all $P \in \operatorname{Spec}(R \otimes_A \Lambda)$, and all $e \geqslant 1$, we have*

$$f_e((M \otimes_A \Lambda)_P) \leqslant C,$$

that is,

$$\ell_{(R\otimes_A\Lambda)_P}\left((M \otimes_A \Lambda)_P / P^{[(\operatorname{char}(\kappa(\mathfrak{p})))^e]}(M \otimes_A \Lambda)_P\right) \leqslant C \cdot (\operatorname{char}(\kappa(\mathfrak{p})))^{e \cdot \dim((M\otimes_A\Lambda)_P)}.$$

Proof. For every Noetherian ring A, finitely generated A-algebra R, and finitely generated R-module M, write

$$\mathscr{C}(A, R, M) = \sup_{\mathfrak{p},\Lambda,P,e} f_e((M \otimes_A \Lambda)_P)$$

where the supremum is taken over all $\mathfrak{p} \in \operatorname{Spec}(A)$ whose residue field $\kappa(\mathfrak{p})$ has positive characteristic, all regular $\kappa(\mathfrak{p})$-algebras Λ, all $P \in \operatorname{Spec}(R \otimes_A \Lambda)$, and all $e \geqslant 1$. Note that, as a regular ring is a product of regular domains, it suffices to restrict to regular $\kappa(\mathfrak{p})$-algebra domains Λ. Moreover, if $\mathfrak{q}_1, \ldots, \mathfrak{q}_l$ are the finitely many minimal primes of A, we have

$$\mathscr{C}(A, R, M) = \max_j \mathscr{C}(A/\mathfrak{q}_j, R/\mathfrak{q}_j R, M/\mathfrak{q}_j M).$$

Thus, it suffices to prove the theorem under the additional assumption that A is a domain.

Suppose by way of contradiction that there is a counterexample for some Noetherian domain A, finitely generated A-algebra R, and finitely generated R-module M. By Noetherian induction (on A), we may assume that the theorem holds for all proper quotient rings $\bar{A}$ of A. In other words, we have $\mathscr{C}(\bar{A}, \bar{R}, \bar{M}) < \infty$ for all finitely generated $\bar{A}$-algebras $\bar{R}$, and all finitely generated $\bar{R}$-modules $\bar{M}$. In particular, for any $0 \neq a \in A$, we have $\mathscr{C}(A/aA, R/aR, M/aM) < \infty$. We will exibit an element $0 \neq a \in A$ for which this fails.

Consider a prime cyclic filtration

$$0 = M_n \subseteq M_{n-1} \subseteq \cdots \subseteq M_i \subseteq \cdots \subseteq M_1 \subseteq M_0 = M.$$

of M, with $Q_i \in \operatorname{Spec}(R)$ satisfying $M_{i-1}/M_i \simeq R/Q_i$ for $i = 1, \ldots, n$. By Theorem 2.12, there is an element $0 \neq a \in A$ such that $R'_i := A/Q_i \otimes_A A[a^{-1}]$ is a free module over $A[a^{-1}]$ for all $i = 1, \ldots, n$. Note that, whenever nonzero, R'_i is thus a finitely generated $A[a^{-1}]$-algebra domain extension of $A[a^{-1}]$. Applying Theorem 2.4 and replacing a with a nonzero multiple, we can ensure that each $R'_i \neq 0$ is module finite over a subring A'_i that is a polynomial ring over $A[b^{-1}]$. In this case, each R'_i is torsion free over A'_i and hence can be realized as a submodule of $(A'_i)^{\oplus m_i}$ where m_i is the torsion free rank of R'_i over A'_i. Applying Theorem 2.12 and replacing a with a nonzero multiple once again, we may assume further that the quotient modules

$K_i := R'_i/A'_i$ and $L_i := (A'_i)^{\oplus m_i}/R'_i$ are free modules over $A[a^{-1}]$ for any i such that $R'_i \neq 0$.

Let us first argue that $\mathscr{C}(A[a^{-1}], R \otimes_A A[a^{-1}], M \otimes_A A[a^{-1}]) < \infty$. Suppose $\mathfrak{p} \in \operatorname{Spec}(A[a^{-1}])$ with residue field $\operatorname{char}(\kappa(\mathfrak{p})) > 0$, Λ is a regular $\kappa(\mathfrak{p})$-algebra domain, $P \in \operatorname{Spec}(R \otimes_A \Lambda)$, and $e \geqslant 1$. As localization is exact, $M_\bullet \otimes_A A[a^{-1}]$ remains a filtration of $M \otimes_A A[a^{-1}]$ with factors R'_i for $i = 1, \ldots, n$. As each R'_i is free over $A[a^{-1}]$, it follows further that $M_\bullet \otimes_A \Lambda = M_\bullet \otimes_A A[a^{-1}] \otimes_{A[a^{-1}]} \Lambda$ remains a filtration of $M \otimes_A \Lambda = M \otimes_A A[a^{-1}] \otimes_{A[a^{-1}]} \Lambda$. By Lemma 2.7, we have that

$$f_e((M \otimes_A \Lambda)_P) \leqslant \sum_{i=1}^{n} f_e((R'_i \otimes_{A[a^{-1}]} \Lambda)_P).$$

For any i such that $R'_i \neq 0$, we have that $A'_i \otimes_{A[a^{-1}]} \Lambda \subseteq R'_i \otimes_{A[a^{-1}]} \Lambda$ remains an extension as K_i is free over $A[a^{-1}]$. Moreover, as L_i is free over $A[a^{-1}]$, $R'_i \otimes_{A[a^{-1}]} \Lambda$ remains a submodule of a free $(A'_i \otimes_{A[a^{-1}]} \Lambda)$-module and so is torsion free over $A'_i \otimes_{A[a^{-1}]} \Lambda$. Now, $A'_i \otimes_{A[a^{-1}]} \Lambda$ is a polynomial ring over Λ, so in particular a regular domain. By Lemma 2.5, it follows that

$$f_e((R'_i \otimes_{A[a^{-1}]} \Lambda)_P) \leqslant \mu_{A'_i \otimes_{A[a^{-1}]} \Lambda}(R'_i \otimes_{A[a^{-1}]} \Lambda) \leqslant \mu_{A'_i}(R'_i)$$

and we conclude $\mathscr{C}(A[a^{-1}], R \otimes_A A[a^{-1}], M \otimes_A A[a^{-1}]) \leqslant \sum_{i=1}^{n} \mu_{A'_i}(R'_i) < \infty$.

To conclude, note that we have

$$\mathscr{C}(A, R, M) = \max\{\mathscr{C}(A/aA, R/aR, M/aM), \mathscr{C}(A[a^{-1}], R \otimes_A A[a^{-1}], M \otimes_A A[a^{-1}])\},$$

and since $\mathscr{C}(A, R, M) = \infty$ we must also have $\mathscr{C}(A/aA, R/aR, M/aM) = \infty$ which is a contradiction. □

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