# Hypergeometric systems I: motivation
# or
# What Gauss knew of variations of Hodge structures

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# Outline

1. Elliptic curves
2. Elliptic integrals

# Elliptic curves 1

- In $\mathbb{C}^2$ consider, with $t \in \mathbb{C}$,

$$E_t = \mathrm{Var}(y^2 - \underbrace{x(x-1)(x-t)}_{=: f_t(x)}).$$

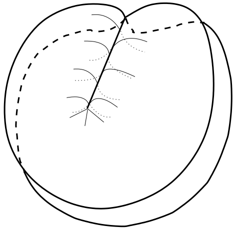
- Singular locus: $y^2 - f_t(x) = 0 = 2y = f_t'(x)$.
  For $t \notin \{0, 1, \infty\}$, $f_t$ and $f_t'$ have no common zero.
- $E_t$ has closure $\bar{E}_t \subseteq \mathbb{P}^2_{\mathbb{C}}$ defined by
  $Y^2 Z - X(X - Z)(X - tZ) = 0$.
- $\bar{E}_t \cap \infty = \mathrm{Var}(Y^2 Z - X(X - Z)(X - tZ), Z) = \mathrm{Var}(X^3, Z)$.
- In $y \neq 0$, $\bar{E}_t$ given by $z' - x'(x' - z')(x' - tz') = 0$.
  Elementary analysis: $z' \approx x'^3$ near $(0, 0)$, so $\bar{E}_t$ smooth.

# Elliptic curves II



$$E_t = \mathrm{Var}(y^2 - \underbrace{x(x-1)(x-t)}_{=:\, f_t(x)}).$$

graph near $0, 1, t, \infty$:



Consider

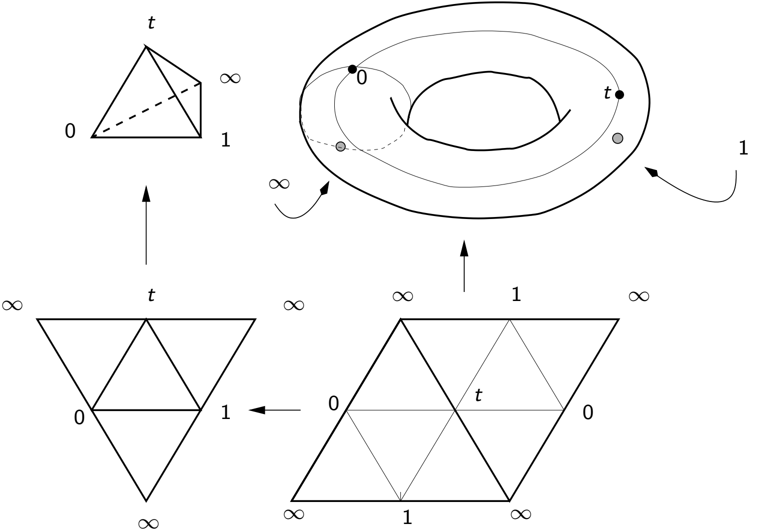
- $$\begin{array}{rcl} E_t & \to & \mathbb{C}, \\ (x, y) & \mapsto & (x) \end{array}$$

  (generically $2:1$).

- As a Riemann surface:
  $E_t$ is a double cover over $\mathbb{C}$, branched at $0, 1, t$.
  $\bar{E}_t$ is a double cover over $\mathbb{P}^1_{\mathbb{C}}$, also branched at $\infty$.

# Elliptic curves II 1/2: global picture






In particular, $\bar{E}_t \cong \mathbb{S}^1 \times \mathbb{S}^1$.

# Differential forms on $\mathrm{Var}(y^2 - x(x-1)(x-t))$ 4

$E_t = \mathrm{Var}(y^2 - f_t(x))$.

- Special 1-form on $E_t$: $\omega_t = dx/y = dx/\sqrt{f_t(x)} = 2dy/f_t'(x)$.
- If $f_t(x) = 0$ then $f_t'(x) \neq 0$, so $\omega_t$ global on $E_t$.
- at infinity: in coordinates where $y \neq 0$, $\omega$ is $dx' - x'dz'/z'$. But, near $(0,0)$: $z' \approx x'^3$, hence $dz' \approx 3x'^2 dx'$ and so

$$dx' - x'dz'/z' \approx dx'(1 - 3x'x'^2/x'^3) = -2dx'$$

  is form of "first kind" (global).
- Conclusion: $\bar{E}_t$ compact complex manifold with global nonvanishing 1-form. "Calabi-Yau curve". (In higher dim, also require simply connected.)

# Integration on $\bar{E}_t = \mathrm{Var}(y^2 - x(x-1)(x-t))$ 

- Form of "second kind": $\omega' = x(x-1)dx/2y^3 = \omega/(x-t)$.
- Near $x = t$: $\omega' = \omega/(x-t) = 2dy/f' \cdot (x-t) = 2dy/y^2$ integrable.
- $\lambda_1, \lambda_2$ generators of $H_1(\bar{E}_t) \cong \mathbb{Z} \oplus \mathbb{Z}$.
- Periods:

$$I_{1,1} = \int_{\lambda_1} \omega, \qquad I_{2,1} = \int_{\lambda_2} \omega, \qquad I_{1,2} = \int_{\lambda_1} \omega', \qquad I_{2,2} = \int_{\lambda_2} \omega'.$$

- These are elliptic integrals, multivalued functions on $\bar{E}_t$.

(Forms of third kind have residue near pole, like $dx/x$.)

# Integration II: periods as a family



- $\frac{d}{dt}(\omega) = \omega'$, $\qquad \frac{d}{dt}(\omega') = \frac{d}{dt}(\frac{\omega}{x-t}) = 2\frac{\omega'}{x-t}$ (product rule).
- Better:
$$\frac{d}{dt}(\omega') = \underbrace{\frac{1}{4t(1-t)}}_{p(t)}\omega + \underbrace{\frac{-1+2t}{t(1-t)}}_{q(t)}\omega' + d\left(\frac{y}{2(x-t)^2 t(1-t)}\right).$$
- Now $\lambda$ any path on $E_t$, set $I_1(\lambda) = \int_\lambda \omega$, $I_2(\lambda) = \int_\lambda \omega'$. Then
$$\begin{aligned} \frac{d}{dt} I_1(\lambda) &= I_2(\lambda) \\ \frac{d}{dt} I_2(\lambda) &= pI_1(\lambda) + qI_1(\lambda). \end{aligned}$$
- The $\omega$-integrals $I_{1,1} = I_1(\lambda_1)$ and $I_{1,2} = I_1(\lambda_2)$ are solutions to
$$z'' - qz' - pz = 0.$$

# Integration III: more on periods



- General hypergemetric diffeq:

$$z'' + \frac{c-(a+b+1)t}{t(1-t)}z' - \frac{ab}{t(1-t)}z = 0. \tag{1}$$

$a = 1/2 = b,\ c = 1.$

- Solutions:

$$\begin{aligned} F_1 &= \sum_{n=0}^{\infty} \frac{[a]_n[b]_n}{[c]_n}\frac{t^n}{n!} \\ F_2 &= -i \cdot \sum_{n=0}^{\infty} \frac{[a]_n[b]_n}{[c]_n}\frac{(1-t)^n}{n!}. \end{aligned}$$

- Which linear comb gives the $\omega$-integrals for $\lambda = \lambda_1, \lambda_2$?

# Integration IV: monodromy from solutions



Monodromy from solutions:
how do solutions vary when $t$ varies?

- Fix the basis $F = (F_1, F_2)$ for (1) in a generic point. Analytic continuation along any loop $\lambda \in \pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\})$ gives $M_\lambda \cdot F$, $M_\lambda \in \mathrm{Gl}(2, \mathbb{C})$.
- Around $0 \in \mathbb{C}$: $M = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$,

  Around $1 \in \mathbb{C}$: $M = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

# Integration V: monodromy from deforming curves



Monodromy from moving curves:
how do integrals vary when we look at the $\bar{E}_t$ as a family?

Note: $y^2 - x(x-1)(x-t) = (y^2 - x^3 - x^2) - t(x(x-1))$.

- $\bar{E}_t \cong \mathbb{S}^1 \times \mathbb{S}^1$ with homology generators $\lambda_1, \lambda_2$.
- Consider $\pi\colon \mathbb{P}^2_{\mathbb{C}} \setminus \{(1,0,0),(1,1,0),(0,0,1)\} \to \mathbb{P}^1_{\mathbb{C}}$, $t_0 = zx(x-z)$, $t_1 = zy^2 - x^3 - x^2 z$. Then $\pi^{-1}(t_0/t_1) = \bar{E}_{t_0/t_1}$.
- Locally in $t$, $\pi$ is a projection $E \times U \to U$, and identifies neighboring $H_1(\bar{E}_t)$.
- The loops $\lambda_1, \lambda_2$ generate $H_1(\bar{E}_t)$.
- Induces $H_1(\bar{E}_t) \to H_1(\bar{E}_t)$, given by $2 \times 2$-matrix $M$.

# Integration VI: comparing monodromies



- Lengthy computation:
  around $t = 0$, $M = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$, around $t = 1$, $M = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.
- we find: $I_{1,1} \leftrightarrow F_1$, $\quad I_{1,2} \leftrightarrow F_2$ up to scalars.
- Periods are hypergeometric functions.

Reference: Brieskorn/Knörrer, "Ebene algebraische Kurven"

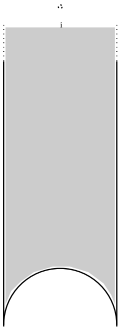
# Afterthought: classifying elliptic curves 

$$\pi : \mathbb{P}^1_{\mathbb{C}} \setminus \{0,1,\infty\} \ni t \mapsto \left(\int_{\lambda_1} \omega_t, \int_{\lambda_2} \omega_t\right) \in \mathbb{P}^1_{\mathbb{C}}.$$

- Let $\pi(t) = \tau_t$, consider $\Lambda_t = \mathbb{Z} + \mathbb{Z}\tau_t \subseteq \mathbb{C}$.
- Let $Q_t = \mathbb{C}/\Lambda_t$, $\tau_t$ the “modulus”. ($\rightsquigarrow$ :moduli spaces”)
- When is $Q_t \cong Q_{t'}$?

If $(I_{1,1}(t), I_{1,2}(t)) = M \cdot (I_{1,1}(t), I_{1,2}(t))$
with $M \in GL(2,\mathbb{Z})$.

Fundamental domain:



# Afterthought: classifying elliptic curves



On $Q_t$ have Weierstraß $\mathfrak{P}$-function

$$\mathfrak{P}(z,\tau_t) = \frac{1}{z^2} + \sum_{m+2+n^2\neq 0} \left\{ \frac{1}{(z-m-n\tau_t)^2} - \frac{1}{(m+n\tau_t)^2} \right\}.$$

$\mathfrak{P}$, $\frac{d}{dt}(\mathfrak{P})$ behave like $x, y$ on $E_t$ if $\tau = I_{1,1}(t)/I_{1,2}(t)$.