INTRODUCTION TO LOCAL COHOMOLOGY AND FROBENIUS

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Throughout this lecture, p denotes a prime number and R denotes a commutative noetherian ring of characteristic p.

Definition 0.1. The *Frobenius endomorphism* of R is the map

$$f : R \xrightarrow{r \mapsto r^p} R.$$

Since R has characteristic p, it follows from the binomial formula that $(a+b)^p = a^p + b^p$ and consequently f is a ring homomorphism, and so are its iterates f^e for each integer $e \geq 1$.

We will use $F^e_* R$ to denote the R-module that is the same as R as an abelian group and whose R-module structure is defined via $F^e : R \to R$. The Peskine-Szpiro functor, denoted by F_R, is defined via

$$F_R(M) := F_* R \otimes_R M.$$

The e-th iteration of F_R is denoted by F^e_R, which is clearly given by $F^e_R(M) = F^e_* R \otimes_R M$.

Exercise 0.2. Prove that

(1) $F^e_R(R^\ell) \cong R^\ell$ for all free modules R^ℓ;
(2) $F^e_R(R/I) \cong R/I^{[p^e]}$ for each ideal I of R;
(3) $F^e_R(R_g) \cong R_g$ for each element $g \in R$;
(4) F_R commutes with localization, *i.e*, $F_R(M)_{\mathfrak{p}} \cong F_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$.

Let $\mathfrak{a} = (g_1, \ldots, g_n)$ be an ideal of R and $\varphi : R \to S$ be a ring homomorphism between commutative noetherian rings of characteristic p. Tensoring φ with the Čech complex $\check{C}(R; \underline{g})$ produces a map of complexes:

$$\check{C}(R; \underline{g}) \to \check{C}(S; \varphi(\underline{g})).$$

Therefore, there is a homomorphism of R-modules

$$H^i_{\mathfrak{a}}(R) \to H^i_{\varphi(\mathfrak{a})S}(S).$$

Specializing in the case when $S = R$ and $\varphi = f$, we have the *Frobenius action* on $H^i_{\mathfrak{a}}(R)$:

$$H^i_{\mathfrak{a}}(R) \to H^i_{f(\mathfrak{a})R}(R) = H^i_{\mathfrak{a}}(R),$$

where the equality follows from $f(\mathfrak{a}) = (g_1^p, \ldots, g_n^p)$ which has the same radical as $\mathfrak{a}$. We will denote this Frobenius action by F.

Example 0.3. Let $R = k[x_1, \ldots, x_d]$ (or $R = k[[x_1, \ldots, x_d]]$) and $\mathfrak{a} = (x_1, \ldots, x_d)$ where k is a field of characteristic p. The Frobenius action on $H^d_{\mathfrak{a}}(R)$ is given by

$$[\frac{c}{x_1^{e_1} \cdots x_d^{e_d}}] \mapsto [\frac{c^p}{x_1^{pe_1} \cdots x_d^{pe_d}}],$$

where $c \in k$.

Example 0.4. Let $(R, \mathfrak{m})$ be a noetherian local ring and $x_1, \dots, x_d$ be a system of parameters, where $d = \dim(R)$. Then each element of $H^d_{\mathfrak{m}}(R)$ can be written as $[\frac{r}{x_1^{e_1} \cdots x_d^{e_d}}]$ for some $r \in R$ and positive integers e_i. The Frobenius action on $H^d_{\mathfrak{m}}(R)$ is given by

$$[\frac{r}{x_1^{e_1} \cdots x_d^{e_d}}] \mapsto [\frac{r^p}{x_1^{pe_1} \cdots x_d^{pe_d}}].$$

One may notice that, in the case when $R = k[x_1, \dots, x_d]$, one has R is a free module over $f(R) = k^p[x_1^p, \dots, x_d^p]$; hence $f : R \to R$ is flat. More generally, we have the following theorem due to Kunz [Kun69].

Theorem 0.5 (Kunz)**.** *The following are equivalent:*

(1) *R is regular;*
(2) *f^e is a flat ring homomorphism for each $e \geq 1$;*
(3) *f^e is a flat ring homomorphism for some $e \geq 1$.*

Theorem 0.6. *Assume that R is regular and $\mathfrak{a}$ is an ideal of R. Then*

$$F^e_R(H^i_{\mathfrak{a}}(R)) \cong H^i_{\mathfrak{a}}(R).$$

Proof. Since $H^i_{\mathfrak{a}}(R) = \varinjlim_n \operatorname{Ext}^i(R/\mathfrak{a}^n, R)$, we have

$$\begin{aligned} F^e_* R \otimes_R H^i_{\mathfrak{a}}(R) &= F^e_* R \otimes_R \varinjlim_n \operatorname{Ext}^i(R/\mathfrak{a}^n, R) \\ &\cong \varinjlim_n F^e_* R \otimes_R \operatorname{Ext}^i(R/\mathfrak{a}^n, R) \\ &\cong \varinjlim_n \operatorname{Ext}^i(F^e_* R \otimes_R R/\mathfrak{a}^n, F^e_* R \otimes_R R) \text{ since } F^e_* R \text{ is flat} \\ &\cong \varinjlim_n \operatorname{Ext}^i(R/(\mathfrak{a}^{[p^e]})^n, R) \text{ by Exercise 0.2} \\ &\cong H^i_{\mathfrak{a}^{[p^e]}}(R) \\ &= H^i_{\mathfrak{a}}(R) \end{aligned}$$

Alternatively, one may also prove it via the Čech complex characterization of $H^i_{\mathfrak{a}}(R)$, using the facts that $F^e_* R$ is flat and that $F^e_R(R_g) \cong R_g$. □

Exercise 0.7. Prove that, if R is regular, then $F_R(E) \cong E$ for each injective R-module E.

1. Vanishing of local cohomology modules

Exercise 1.1. Let $(R, \mathfrak{m})$ be a regular local ring of characteristic p. Assume that $\operatorname{Ann}_R(H^i_{\mathfrak{a}}(R)) \neq 0$. Prove $H^i_{\mathfrak{a}}(R) = 0$.

Exercise 1.2. Let $\mathfrak{p}$ be a prime ideal of a regular ring R of characteristic p. Prove that $\mathfrak{p}^{[p^e]}$ is $\mathfrak{p}$-primary.

Theorem 1.3 (Peskine-Szpiro [PS73])**.** *Let $(R, \mathfrak{m})$ be a regular local ring of characteristic p of dimension n and let I be an ideal of R. Assume that* $\operatorname{depth}(R/I) \geq c$. *Then*

$$H^{n-i}_I(R) = 0 \text{ for } i \leq c - 1.$$

Proof. Since $\operatorname{depth}(R/I) \geq c$, we have $H^i_{\mathfrak{m}}(R/I) = 0$ for $i \leq c-1$. By local duality, $\operatorname{Ext}^{n-i}_R(R/I, R) = 0$. By Theorem 0.5, we have $\operatorname{Ext}^{n-i}_R(R/I^{[p^e]}, R) \cong F^e_* R \otimes_R \operatorname{Ext}^{n-i}_R(R/I, R) = 0$. Hence

$$H^{n-i}_I(R) = \varinjlim_e \operatorname{Ext}^{n-i}_R(R/I^{[p^e]}, R) = 0.$$

□

Example 1.4. Let $R = k[[x_{ij}]]$ with $i = 1, 2$ and $j = 1, 2, 3$ where k is a field. Let I be generated by the 2×2 minors of (x_{ij}). Then R/I has depth 4 by [HE71].

When k has characteristic p, by Theorem 1.3, we have $H^i_I(R) \neq 0$ if and only if $i = 2$.

In contrast, when k has characteristic 0, one has $H^3_I(R) \neq 0$ ([HS77, page 75]).

Exercise 1.5. Let $(R, \mathfrak{m})$ be a complete regular local ring and E be the injective hull of the residue field $R/\mathfrak{m}$. Denote the Matlis dual functor $\operatorname{Hom}_R(-, E)$ by $D(-)$. Prove that there is a functorial R-module isomorphism

$$\tau : D(F_R(M)) \cong F_R(D(M))$$

for all artinian R-modules M.

Let A be a noetherian commutative ring of characteristic p. Let $A\{f\}$ be the subring of $\operatorname{Hom}_{\mathbb{Z}}(A, A)$ generated by multiplications by elements of A and $f : A \xrightarrow{a \mapsto a^p} A$, or equivalently $A\{f\} = \frac{A\langle f \rangle}{(fa - a^p f)}$. An $A\{f\}$-module M is an A-module M equipped with a $\mathbb{Z}$-linear map $f : M \to M$ such that $f(am) = a^p f(m)$ for all $a \in A$ and $m \in M$.

Let $(R, \mathfrak{m})$ be a complete regular local ring and A be a homomorphic image of R. Let M be an $A\{f\}$-module. One can check that

$$\alpha : F_R(M) \xrightarrow{r \otimes m \mapsto r f(m)} M$$

is an R-module homomorphism. Now, assume that M is artinian. Taking the Matlis dual of α and applying Exercise 1.5, we have an R-module homomorphism

$$\beta = \tau \circ D(\alpha) : D(M) \to F_R(D(M)),$$

and hence we have a direct system of R-modules:

$$D(M) \xrightarrow{\beta} F_R(D(M)) \xrightarrow{F_R(\beta)} F^2_R(D(M)) \to \cdots$$

Definition 1.6 (Lyubeznik's $\mathscr{H}_{R,A}$ functor [Lyu97])**.** Let $(R, \mathfrak{m})$ be a complete regular local ring and A be a homomorphic image of R. For each artinian $A\{f\}$-module, we define

$$\mathscr{H}_{R,A}(M) := \varinjlim (D(M) \xrightarrow{\beta} F_R(D(M)) \xrightarrow{F_R(\beta)} F^2_R(D(M)) \to \cdots)$$

Example 1.7. Let $(R, \mathfrak{m})$ be a complete regular local ring of dimension n and A be a homomorphic image of R. Let I be the kernel of $R \twoheadrightarrow A$. From our discussion, we know that $H^i_{\mathfrak{m}}(A)$ is naturally an $A\{f\}$-module. We would like to understand $\mathscr{H}_{R,A}(H^i_{\mathfrak{m}}(A))$.

By local duality, we have $D(H^i_{\mathfrak{m}}(A)) \cong \operatorname{Ext}^{n-i}_R(R/I, R)$ and hence the direct system

$$\operatorname{Ext}^{n-i}_R(R/I, R) \xrightarrow{\beta} F_R(\operatorname{Ext}^{n-i}_R(R/I, R)) \xrightarrow{F_R(\beta)} F^2_R(\operatorname{Ext}^{n-i}_R(R/I, R)) \to \cdots$$

which is the same as

$$\operatorname{Ext}^{n-i}_R(R/I, R) \to \operatorname{Ext}^{n-i}_R(R/I^{[p]}, R) \to \operatorname{Ext}^{n-i}_R(R/I^{[p^2]}, R) \to \cdots$$

whose direct limit is $H^{n-i}_I(R)$. Therefore,

$$\mathscr{H}_{R,A}(H^i_{\mathfrak{m}}(A)) \cong H^{n-i}_I(R).$$

Theorem 1.8 (Theorem 1.1 in [Lyu06]). *Let $(R,\mathfrak{m})$ be a complete regular local ring of dimension n and A be a homomorphic image of R. Let I be the kernel of $R \twoheadrightarrow A$. Then $H^{n-i}_I(R) = 0$ if and only if the Frobenius action on $H^i_{\mathfrak{m}}(A)$ is nilpotent.*

Proof. First note that, since

$$H^{n-i}_i(R) = \varinjlim_e \operatorname{Ext}^{n-i}_R(R/I^{[p^e]}, R)$$

and

$$F^e_R(\operatorname{Ext}^{n-i}_R(R/I,R)) \cong \operatorname{Ext}^{n-i}_R(R/I^{[p^e]}, R),$$

we see that $H^{n-i}_i(R) = 0$ if and only if there is t such that

$$\varphi_t : \operatorname{Ext}^{n-i}_R(R/I,R) \to \operatorname{Ext}^{n-i}_R(R/I^{[p^t]}, R)$$

is the 0 map.

With notation as in Example 1.7, $\varphi_t = 0$ if and only if $F^{t-1}_R(\beta) \circ \cdots F_R(\beta) \circ \beta = 0$. Since $\operatorname{Ext}^{n-i}_R(R/I,R)$ is finitely generated (or equivalently $H^i_{\mathfrak{m}}(R/I)$ is artinian), by Matlis duality, $F^{t-1}_R(\beta) \circ \cdots F_R(\beta) \circ \beta = 0$ if and only if the following composition of maps is 0

$$F^t_R(H^i_{\mathfrak{m}}(R/I)) \xrightarrow{F^{t-1}_R(\alpha)} F^{t-1}_R(H^i_{\mathfrak{m}}(R/I)) \to \cdots \to F_R(H^i_{\mathfrak{m}}(R/I)) \xrightarrow{\alpha} H^i_{\mathfrak{m}}(R/I).$$

Writing $F^t_R(H^i_{\mathfrak{m}}(R/I))$ as $F_*R \otimes_R \cdots \otimes_R F_*R \otimes_R H^i_{\mathfrak{m}}(R/I)$, one can check that

$$\alpha \circ \cdots \circ F^{t-1}_R(\alpha)(r_t \otimes \cdots \otimes r_1 \otimes z) = r_t r^p_{t-1} \cdots r^{p^t}_1 f^t(z)$$

for all $r_i \in F_*R$ and $z \in H^i_{\mathfrak{m}}(R/I)$. It is now straightforward to check that $\alpha \circ \cdots \circ F^{t-1}_R(\alpha) = 0$ if and only if $f^t = 0$, *i.e.* the Frobenius action $f : H^i_{\mathfrak{m}}(R/I) \to H^i_{\mathfrak{m}}(R/I)$ is nilpotent. This completes the proof. ☐

In light of Theorem 1.8, we introduce the notion of F-depth:

Definition 1.9. Let $(A,\mathfrak{m})$ be a noetherian local ring of characteristic p. The F-depth of A is the smallest integer j such that the Frobenius on $H^j_{\mathfrak{m}}(A)$ is nilpotent.

Exercise 1.10. Prove that $F\text{-depth}(A) \le \dim(A)$.

As an application of Theorem 1.8, we will state (without proofs) a result on cohomological dimension of an open subset of $\mathbb{P}^n_k$, where k is a field of characteristic p. To this end, let's recall the definition of cohomological dimension.

Definition 1.11. Let $Y \subset \mathbb{P}^n_k$ be a closed subscheme and U be the complement of Y in $\mathbb{P}^n_k$. The cohomological dimension of U, denoted by $\operatorname{cd}(U)$, is the largest integer i such there is a quasi-coherent sheaf $\mathcal{F}$ on U such that $H^i(U,\mathcal{F}) \neq 0$.

Remark 1.12. Set $R = k[x_0,\dots,x_n]$ and I to be the homogeneous defining ideal of Y. Then it is proved in [Har68, pp. 412-413] that $\operatorname{cd}(U) < c$ if and only if $H^j_I(R) = 0$ for all $j > c$.

Theorem 1.13 (Corollary 5.4 in [Lyu06]). *Let Y,U be the same as above. For each $c \ge 2$, we have $\operatorname{cd}(U) < n - c$ if and only if the following 3 conditions hold:*

(1) *Y is geometrically connected and* $\dim(Y) > 0$*;*
(2) *F-depth$(\mathcal{O}_{Y,y}) \geq c$ for every closed point $y \in Y$;*
(3) *the Frobenius action on $H^i(Y, \mathcal{O}_Y)$ is nilpotent for each $1 \leq j \leq c-1$.*

2. Associated primes and Bass numbers of local cohomology modules

A question of Huneke [Hun92, Problem 4] asks whether local cohomology modules of Noetherian rings have finitely many associated prime ideals. The answer is negative in general; the first counterexample was given by Singh [Sin00, § 4]. There are positive answers in some cases; for instance, when the ring is a regular ring of characteristic p. we want to discuss the approach in [HS93].

Recall that, when A is a commutative noetherian local ring and M is an A-module, the Bass number $\mu^i(\mathfrak{p}, M)$ (with respect to a prime ideal $\mathfrak{p}$ of A) is defined as

$$\mu^i(\mathfrak{p}, M) := \dim_{\kappa(\mathfrak{p})}(\mathrm{Ext}^i_{A_\mathfrak{p}}(\kappa(\mathfrak{p}), M_\mathfrak{p})),$$

where $\kappa(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p} A_\mathfrak{p}$. Alternatively, let $E^i(M)$ denote the i-th item in the minimal injective resolution of M. One has a decomposition $E^i(M) \cong \oplus_{\lambda\in\Lambda} E(A/\mathfrak{p}_\lambda)$. Then for each prime $\mathfrak{p}$, one has $\mu^i(\mathfrak{p}, M)$ is the cadinality of the set $\{\lambda \in \Lambda \mid \mathfrak{p}_\lambda = \mathfrak{p}\}$.

Exercise 2.1. Prove that $\mathfrak{p} \in \mathrm{Ass}_A(M)$ if and only if $\mu^0(\mathfrak{p}, M) > 0$.

Exercise 2.2. Let $(R, \mathfrak{m})$ be a noetherian local ring and let $\{\mathcal{I}_j\}_{j\in\mathbb{Z}_{>0}}$ be a direct system of injective R-modules with limit $\mathcal{I}$. Assume that there is an integer μ such that each $\mathcal{I}_j$ has at most μ copies of $E(R/\mathfrak{m})$. Prove $\mathcal{I}$ also has at most μ copies of $E(R/\mathfrak{m})$.

Theorem 2.3 (Huneke-Sharp)**.** *Let $(R, \mathfrak{m})$ be a regular local ring of characteristic p and I be an ideal of R. Then $\mu^i(\mathfrak{p}, H^j_I(R)) \leq \mu^i(\mathfrak{p}, \mathrm{Ext}^j_R(R/I, R))$ for each prime ideal $\mathfrak{p}$ and all integers i, j. In particular, $\mu^i(\mathfrak{p}, H^j_I(R))$ is finite.*

Proof. After localizing at $\mathfrak{p}$, we may and we do assume that $\mathfrak{p} = \mathfrak{m}$. Let

$$0 \to \mathrm{Ext}^j_R(R/I, R) \to E^0 \to E^1 \to \cdots$$

be a minimal injective resolution of $\mathrm{Ext}^j_R(R/I, R)$. For each positive integer t, we will denote $F^t_R(E^j)$ by E^j_t. By Exercise 0.7, we know that $E^j_t \cong E^j$ for each t. Applying the exact functor F^t_R to this injective resolution of $\mathrm{Ext}^j_R(R/I, R)$, we get an exact sequence

$$0 \to \mathrm{Ext}^j_R(R/I^{[p^t]}, R) \to E^0_t \to E^1_t \to \cdots$$

which is a minimal injective resolution of $\mathrm{Ext}^j_R(R/I^{[p^t]}, R)$. The functoriality of F_R and standard results on injective resolutions produce a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ext}^j_R(R/I,R) & \longrightarrow & E^0 & \longrightarrow \cdots \longrightarrow & E^j & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Ext}^j_R(R/I^{[p]},R) & \longrightarrow & E^0_1 & \longrightarrow \cdots \longrightarrow & E^j_1 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Ext}^j_R(R/I^{[p^t]},R) & \longrightarrow & E^0_t & \longrightarrow \cdots \longrightarrow & E^j_t & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Taking direct limits and keeping in mind that a direct of injective modules is still injective, we get an exact sequence

$$0 \to H^j_I(R) \to \mathcal{E}^0 \to \mathcal{E}^1 \to \cdots$$

which is a minimal injective resolution of $H^j_I(R)$. It follows from Exercise 2.2 and $E^j_t \cong E^j$ that

$$\mu^i(\mathfrak{p}, H^j_I(R)) \le \mu^i(\mathfrak{p}, \mathrm{Ext}^j_R(R/I, R)).$$

□

As an immediate consequence of Theorem 2.3 and Exercise 2.1, we have:

Theorem 2.4 (Huneke-Sharp)**.** *Let R be a regular ring of characteristic p and I be an ideal of R. Then $\mathrm{Ass}_R(H^j_I(R)) \subseteq \mathrm{Ass}_R(\mathrm{Ext}^j_R(R/I,R))$. In particular, $\mathrm{Ass}_R(H^j_I(R))$ is a finite set.*

There are still many interesting question regarding 'finiteness' properties of local cohomology modules in characteristic p when R is not regular.

Question 2.5. Let S be a regular ring of characteristic p and G be a finite group acting on S with $p \mid |G|$. Set $R = S^G$. Is it true that each local cohomology module $H^j_I(R)$ has finitely many associated primes for an ideal I of R?

Question 2.6. Let R be a Stanley-Reisner ring of characteristic p. Is it true that each local coho mology module $H^j_I(R)$ has finitely many associated primes for an ideal I of R?

When R is Gorenstein, Question 2.6 has a positive answer [TT08, 3.3].

Next, I'd like to discuss Lyubeznik's F-module theory, which is a generalization of what we have seen so far.

3. F-modules

Definition 3.1. Let R be a regular ring of characteristic p. An F-module is an R-module M equipped with an R-linear isomorphism $\theta_M : M \to F_R(M)$.

A homomorphism between F-modules (M, θ_M) and (N, θ_N) is a homomorphism $\varphi : M \to N$ such that the following is a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\theta_M} & F_R(M) \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle F_R(\varphi)} \\ N & \xrightarrow{\theta_N} & F_R(N). \end{array}$$

Example 3.2. It follows from Exercise 0.2 and Theorem 0.6 that R^ℓ, R_g, and $H^j_{\mathfrak{a}}(R)$ are F-modules.

From what we have seen, the morphism $\operatorname{Ext}^j_R(R/\mathfrak{a}, R) \to F_R(\operatorname{Ext}^j_R(R/\mathfrak{a}, R))$ plays a key role in understanding $H^j_{\mathfrak{a}}(R)$. The analog of such a morphism for an F-module is called a generating morphism.

Definition 3.3. Let R be a regular ring of characteristic p and $\mathscr{M}$ be an F-module. A *generating morphism* of $\mathscr{M}$ is an an R-module homomorphism $M \to F_R(M)$, with some R-module M, such that such that $\mathscr{M}$ is the direct of the top row of the following commutative diagram:

$$\begin{array}{ccccccc} M & \xrightarrow{\beta} & F_R(M) & \xrightarrow{F_R(\beta)} & F^2_R(M) & \xrightarrow{F^2_R(\beta)} & \cdots \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle F_R(\beta)} & & \downarrow{\scriptstyle F^2_R(\beta)} & & \\ F_R(M) & \xrightarrow{F_R(\beta)} & F^2_R(M) & \xrightarrow{F^2_R(\beta)} & F^3_R(M) & \xrightarrow{F^3_R(\beta)} & \cdots \end{array}$$

and the structure isomorphism $\theta : \mathscr{M} \to F_R(\mathscr{M})$ is induced from the vertical arrows in the diagram.

Remark 3.4. Since tensor product commutes with direct limits, so the direct limit of the bottom row is indeed $F_R(\mathscr{M})$.

It is clear that each F-module $\mathscr{M}$ has a generating morphism, the structure isomorphism.

Example 3.5. The morphism $\operatorname{Ext}^j_R(R/\mathfrak{a}, R) \to F_R(\operatorname{Ext}^j_R(R/\mathfrak{a}, R))$ is clearly a generating morphism for the F-module $H^j_{\mathfrak{a}}(R)$.

Given $f \in R$, there is a map of Koszul complexes

$$\begin{array}{ccccccccc} K^\bullet(f; R) & = & 0 & \longrightarrow & R & \xrightarrow{f} & R & \longrightarrow & 0 \\ \downarrow & & & & \| & & \downarrow{\scriptstyle f^{p-1}} & & \\ K^\bullet(f^p; R) & = & 0 & \longrightarrow & R & \xrightarrow{f^p} & R & \longrightarrow & 0 \end{array}$$

where $K^\bullet$ denotes the Koszul complex. Let $\boldsymbol{f} = f_1, \ldots, f_n$ be a sequence of elements of R. Regarding $K^\bullet(\boldsymbol{f}; R)$ as the tensor products

$$K^\bullet(f_1; R) \otimes \cdots \otimes K^\bullet(f_n; R),$$

one obtains a map of complexes

$$K^\bullet(\boldsymbol{f};R) \longrightarrow K^\bullet(\boldsymbol{f}^p;R)\,,$$

and induced maps on cohomology modules

$$H^j(\boldsymbol{f};R) \xrightarrow{\ \beta\ } H^j(\boldsymbol{f}^p;R)\,,$$

where $\mathfrak{a}$ is the ideal generated by $\boldsymbol{f}$. Since $H^j_{\mathfrak{a}}(R)$ is the direct limit of $\{H^j(\boldsymbol{f}^{p^t};R)\}_t$ and $H^j(\boldsymbol{f};R) \cong F_R(H^j(\boldsymbol{f}^p;R))$, the composition $H^j(\boldsymbol{f};R) \to H^j(\boldsymbol{f}^p;R) \cong F_R(H^j(\boldsymbol{f}^p;R))$ is also a generating morphism of $H^j_{\mathfrak{a}}(R)$.

Note that $H^j_{\mathfrak{a}}(R)$ is 'generated' by a finitely generated R-module $\operatorname{Ext}^j_R(R/\mathfrak{a},R)$ (or $H^j(\boldsymbol{f};R)$) via the generating morphism $\operatorname{Ext}^j_R(R/\mathfrak{a},R) \to F_R(\operatorname{Ext}^j_R(R/\mathfrak{a},R))$ (or $H^j(\boldsymbol{f};R) \to H^j(\boldsymbol{f}^p;R)$). This leads to:

Definition 3.6. An F-module $\mathscr{M}$ is called *F-finite* if $\mathscr{M}$ admits a generating morphism $\beta : M \to F_R(M)$ with M a finitely generated R-module.

If, in addition, β is injective, M (or $\beta(M)$) is called a *root* of $\mathscr{M}$ and β is called a *root morphism.*

Theorem 3.7. *Every F-finite module has a root.*

Proof. Let $\mathscr{M}$ be an F-finite module with a generating morphism $\beta : M \to F_R(M)$ with M a finitely generated R-module. Let $\beta_i : M \to F^i_R(M)$ denote the composition $M \xrightarrow{\beta} F_R(M) \xrightarrow{F_R(\beta)} \cdots \xrightarrow{F_R^{i-1}(\beta)} F^i_R(M)$. It is clear that $\ker(\beta_i) \subseteq \ker(\beta_{i+1})$ and hence $\ker(\beta_i) = \ker(\beta_{i+1}) = \cdots$ for $i \gg 0$.

We claim that $\operatorname{Im}(\beta_i)$ is a root of $\mathscr{M}$.

Let f denote the composition of $\operatorname{Im}(\beta_i) \xrightarrow{\sim} \operatorname{Im}(\beta_{i+1})$ and $\operatorname{Im}(\beta_{i+1}) \hookrightarrow F_R(\operatorname{Im}(\beta_i))$. Since F_R is exact, we have $\operatorname{Im}(F^t_R(\beta_i)) = F^t_R(\operatorname{Im}(\beta_i))$. Hence the direct system $M \xrightarrow{\beta} F_R(M) \xrightarrow{F_R(\beta)} \cdots$ induces a direct system

$$\operatorname{Im}(\beta_i) \xrightarrow{f} F_R(\operatorname{Im}(\beta_i)) \xrightarrow{F_R(f)} F^2_R(\operatorname{Im}(\beta_i)) \xrightarrow{F^2_R(f)} \cdots$$

Then we have a commutative diagram of direct systems

$$\begin{array}{ccccccc}
M & \xrightarrow{\beta} & F_R(M) & \xrightarrow{F_R(\beta)} & F^2_R(M) & \xrightarrow{F^2_R(\beta)} & \cdots \\
\downarrow{\scriptstyle \beta_i} & & \downarrow{\scriptstyle F_R(\beta_i)} & & \downarrow{\scriptstyle F^2_R(\beta_i)} & & \\
\operatorname{Im}(\beta_i) & \xrightarrow{f} & F_R(\operatorname{Im}(\beta_i)) & \xrightarrow{F_R(f)} & F^2_R(\operatorname{Im}(\beta_i)) & \xrightarrow{F^2_R(f)} & \cdots
\end{array}$$

Since vertical maps are surjective, the induced map on the direct limits is surjective. The map on the direct limits is also injective since the kernel of each vertical map eventually goes to 0 in the direct system. Hence the direct limit of the bottom system is also $\mathscr{M}$. Since $\operatorname{Im}(\beta_i)$ is a finite R-module and f is injective, $\operatorname{Im}(\beta_i)$ is a root of $\mathscr{M}$. $\square$

Example 3.8. $H^j_{\mathfrak{a}}(R)$ has a root since it is an F-finite module.

Exercise 3.9. Assume that R is a regular local ring of characteristic p. Let $\mathscr{M}$ be an F-module that is also an artinian R-module. Prove that $\mathscr{M}$ is an injective R-module. (Hint: use the fact that all Bass numbers of $\mathscr{M}$ are finite and consider a similar diagram as in the proof of Theorem 2.3.)

More generally, one has the following.

Theorem 3.10 (Theorem 1.4 in [Lyu97])**.** $\operatorname{inj.dim}_R(\mathscr{M}) \leq \dim_R(\operatorname{Supp}_R(\mathscr{M}))$ *for each F-module $\mathscr{M}$.*

Remark 3.11. An immediate consequence of Theorem 3.10 is that, if $\mathfrak{m}$ is a maximal ideal of R, then $H^i_{\mathfrak{m}}(H^j_{\mathfrak{a}}(R))$ (or more generally, $H^i_{\mathfrak{m}} H^{j_1}_{\mathfrak{a}_1} \cdots H^{j_s}_{\mathfrak{a}_s}(R)$) is a direct sum of some copies of $E(R/\mathfrak{m})$. This leaves the following question open: let $R = k[x_1, \dots, x_n]$ be the polynomial ring over a field k with the usual grading (*i.e.* $\deg(x_i) = 1$) and $\mathfrak{a}_1, \dots, \mathfrak{a}_s$ be graded ideals of R. Set $\mathfrak{m} = (x_1, \dots, x_n)$. Is it true that $H^i_{\mathfrak{m}} H^{j_1}_{\mathfrak{a}_1} \cdots H^{j_s}_{\mathfrak{a}_s}(R) \cong E(R/\mathfrak{m})^{\oplus t}$ for some integer t as graded Rmodules? Or, equivalently, are all the socle elements of $H^i_{\mathfrak{m}} H^{j_1}_{\mathfrak{a}_1} \cdots H^{j_s}_{\mathfrak{a}_s}(R)$ have degree $-n$? We will come back to this question when we discuss $\mathcal{D}$-modules.

The following is a generalization of Theorem 2.3.

Theorem 3.12 (Theorem 2.11 in [Lyu97])**.** *Let $\mathscr{M}$ be an F_R-finite module and M be a root of $\mathscr{M}$. Then*

$$\mu^j(\mathfrak{p}, \mathscr{M}) \leq \mu^j(\mathfrak{p}, M)$$

for each prime ideal $\mathfrak{p}$ and each integer j.

Exercise 3.13. Let $\mathfrak{p}$ be a prime ideal of R. Assume that M is an R-module such that $H^j_{\mathfrak{p}}(M)_{\mathfrak{p}}$ are injective for all j. Prove that

$$\mu^j(\mathfrak{p}, M) = \mu^0(\mathfrak{p}, H^j_{\mathfrak{p}}(M)).$$

Exercise 3.14. Prove Theorem 3.12.

Surprisingly, local cohomology modules, though may not be finitely generated or artinian as R-modules, will have finite length in the category of F_R-modules.

Theorem 3.15. *If R is a noetherian regular ring essentially of finite type over a regular local ring of characteristic p, then every F_R-finite module has finite length in the category of F_R-modules.*

4. $\mathcal{D}$-modules

Let R be a commutative ring. *$\mathbb{Z}$-linear differential operators* on R are defined inductively as follows: for each $r \in R$, the multiplication by r map $\tilde{r}\colon R \to R$ is a differential operator of order 0; for each positive integer n, the differential operators of order less than or equal to n are those additive maps $\delta\colon R \to R$ for which the commutator

$$[\tilde{r}, \delta] \;=\; \tilde{r} \circ \delta - \delta \circ \tilde{r}$$

is a differential operator of order less than or equal to $n - 1$. If δ and δ' are differential operators of order at most m and n respectively, then $\delta \circ \delta'$ is a differential operator of order at most $m + n$. Thus, the differential operators on R form a subring $\mathcal{D}(R)$ of $\operatorname{End}_{\mathbb{Z}}(R)$.

When R is an algebra over a commutative ring A, we define $\mathcal{D}(R, A)$ to be the subring of $\mathcal{D}(R)$ consisting of differential operators that are A-linear. Note that $\mathcal{D}(R, \mathbb{Z}) = \mathcal{D}(R)$.

Example 4.1. Let $R = A[x_1, \dots, x_d]$ or $A[[x_1, \dots, x_d]]$. Then $\frac{1}{t_i!}\frac{\partial^{t_i}}{\partial x_i^{t_i}}$ can be viewed as a differential operator on R even if the integer $t_i!$ is not invertible. In either case, $\mathcal{D}(R, A)$ is the free R-module with basis

$$\frac{1}{t_1!}\frac{\partial^{t_1}}{\partial x_1^{t_1}} \cdots \frac{1}{t_d!}\frac{\partial^{t_d}}{\partial x_d^{t_d}} \qquad \text{for} \quad (t_1, \dots, t_d) \in \mathbb{N}^d,$$

or equivalently $\mathcal{D}(R, A) = R\langle \frac{1}{t_1!}\frac{\partial^{t_1}}{\partial x_1^{t_1}} \cdots \frac{1}{t_d!}\frac{\partial^{t_d}}{\partial x_d^{t_d}} \mid (t_1, \dots, t_d) \in \mathbb{N}^d \rangle$

By the quotient rule, it is clear that R_f is a $\mathcal{D}(R, A)$-module. It is also straightforward to check that the maps in the Čech complex are $\mathcal{D}(R, A)$-linear and hence each local cohomology module $H^i_{\mathfrak{a}}(R)$ is a $\mathcal{D}(R, A)$-module.

When specializing $A = \mathbb{Z}$, we have that $\mathcal{D}(R)/p\,\mathcal{D}(R) = \mathcal{D}(R/pR)$.

Remark 4.2. Let R be a noetherian commutative ring of characteristic p. Assume that R is a finitely generated R^p-module. Then by [Yek92, 1.4.9] and [SVdB97, 2.5.1] we have

$$\mathcal{D}(R) = \mathcal{D}(R, \mathbb{Z}/p\,\mathbb{Z}) = \bigcup_e \operatorname{Hom}_{R^{p^e}}(R, R).$$

Proposition 4.3. *Let R be a regular ring of characteristic p such that R is a finitely generated R^p-module. Then each F_R-module is a $\mathcal{D}(R)$-module.*

Proof. Let $\mathscr{M}$ be an F_R-module with the structure isomorphism $\theta : \mathscr{M} \to F_R(\mathscr{M})$. By Remark 4.2, it suffices to specify the action of each element $\delta \in \operatorname{Hom}_{R^{p^e}}(R, R)$ on $\mathscr{M}$. Let θ_e denote the composition $F^{e-1}(\theta) \circ \cdots \circ \theta : \mathscr{M} \to F^e(\mathscr{M})$. Clearly δ acts on $F^e(\mathscr{M}) = F^e_* R \otimes_R \mathscr{M}$ via $\delta \otimes id_{\mathscr{M}}$. We let δ act on $\mathscr{M}$ via $\theta_e^{-1} \circ (\delta \otimes id_{\mathscr{M}}) \circ \theta_e$.

Since $\delta \in \operatorname{Hom}_{R^{p^e}}(R, R) \subseteq \operatorname{Hom}_{R^{p^{e'}}}(R, R)$ for $e' > e$, we need to check that the action of δ is independent of the choice of e, *i.e.* we need to check that

$$\theta_e^{-1} \circ (\delta \otimes id_{\mathscr{M}}) \circ \theta_e = \theta_{e'}^{-1} \circ (\delta \otimes id_{\mathscr{M}}) \circ \theta_{e'}. \tag{1}$$

Set $\theta^e_{e'-e} = F^e(\theta_{e'-e}) = id_{F^e_* R} \otimes \theta_{e'-e}$. Then $\theta_{e'} = \theta^e_{e'-e} \circ \theta_e$. To check (1) is equivalent to checking

$$\begin{aligned} \delta \otimes id_{\mathscr{M}} &= (\theta^e_{e'-e})^{-1} \circ (\delta \otimes id_{\mathscr{M}}) \circ \theta^e_{e'-e} \\ &= (id_{F^e_* R} \otimes \theta_{e'-e})^{-1} \circ (\delta \otimes id_{\mathscr{M}}) \circ (id_{F^e_* R} \otimes \theta_{e'-e}) \end{aligned}$$

which is clear. ☐

As an application of $\mathcal{D}$-module, now we turn our attention to answering the question raised in Remark 3.11.

Let $R = \mathbb{F}_p[x_1, \dots, x_n]$ and let $\mathcal{D}$ denote the ring of differential operators over R. We will use $\partial_j^{[t]}$ to denote $\frac{1}{t!}\frac{\partial^t}{\partial x_j^t}$.

Definition 4.4. The r-th Euler operator, denoted by E_r, is defined as

$$E_r := \sum_{i_1 + i_2 + \cdots + i_n = r, i_1 \geq 0, \dots, i_n \geq 0} x_1^{i_1} \cdots x_n^{i_n} \partial_1^{[i_1]} \cdots \partial_n^{[i_n]}.$$

In particular E_1 is the usual Euler operator $\sum_{i=1}^n x_i \partial_i$.

A graded $\mathcal{D}$-module M is called *Eulerian*, if each homogeneous element $z \in M$ satisfies

$$E_r \cdot z = \binom{\deg(z)}{r} \cdot z \tag{2}$$

for every $r \geq 1$.

Remark 4.5. An F_R-module $\mathscr{M}$ with structure isomorphism $\theta : \mathscr{M} \to F_R(\mathscr{M})$ is called a *graded F_R-module* if $\mathscr{M}$ is a graded R-module and θ is degree-preserving. It turns out that each graded F_R-module is an Eulerian graded $\mathcal{D}$-module [MZ14, Theorem 4.4].

Exercise 4.6. Prove that, if M is a graded Eulerian $\mathcal{D}$-module, so are $\mathcal{D}$-submodules and $\mathcal{D}$-quotients of M.

Exercise 4.7. Prove that M_f is an Eulerian graded $\mathcal{D}$-module for an Eulerian graded $\mathcal{D}$-module M and a graded element $f \in R$ and . (Hint: $\partial_i^{[j]}$ is p^e-linear for $p^e > j$ and hence $\partial_i^{[j]}(\frac{z}{f^{p^e}}) = \frac{1}{f^{p^e}}\partial_i^{[j]}(z)$ for each $z \in M$.)

Combining Exercises 4.6 and 4.7, we have

Proposition 4.8. *Let $\mathfrak{a}_1, \dots, \mathfrak{a}_s$ be graded ideals of R. Then each local cohomology module $H^i_{\mathfrak{m}} H^{j_1}_{\mathfrak{a}_1} \cdots H^{j_s}_{\mathfrak{a}_s}(R)$ is an Eulerian graded $\mathcal{D}$-module.*

For each graded R-module M and integer ℓ, we will use $M(\ell)$ to denote the graded R-module whose degree-j piece is $M_{j+\ell}$ for each integer j.

Exercise 4.9. Let M be an Eulerian graded $\mathcal{D}$-module. Prove that $M(\ell)$ is Eulerian if and only if $\ell = 0$.

Remark 4.10. The ring of differential operators $\mathcal{D}$ has a natural grading $\deg(x_i) = 1$ and $\deg(\partial_i^{[j]}) = -j$. Under this grading, the map $\frac{\mathcal{D}}{\mathcal{D}\mathfrak{m}} \to H^n_{\mathfrak{m}}(R)(-n)$ defined by

$$\partial_1^{[i_1]} \cdots \partial_n^{[i_n]} \mapsto (-1)^{i_1+\cdots+i_n} x_1^{-i_1-1} \cdots x_n^{-i_n-1}$$

is a degree-preserving isomorphism.

Theorem 4.11. *If M is an Eulerian graded $\mathcal{D}$-module and* $\mathrm{Supp}_R(M) = \{\mathfrak{m}\}$*, then M is isomorphic (as graded R-modules) to a direct sum of copies of $H^n_{\mathfrak{m}}(R)$.*

Proof. Let $\mathrm{Soc}(M)$ denote the socle of M, *i.e* the sub-R-module of M that is annihilated by $\mathfrak{m}$. It is straightforward to check that $\mathrm{Soc}(M)$ is generated by graded elements and a minimal set of graded generators of $\mathrm{Soc}(M)$ is a graded k-basis for $\mathrm{Soc}(M)$.

Let $\{\beta_j\}$ be a graded k-basis of $\mathrm{Soc}(M)$ with $\deg(\beta_j) = -n_j$. There is a degree-preserving homomorphism of $\mathcal{D}$-modules $\bigoplus_j \frac{\mathcal{D}}{\mathcal{D}\mathfrak{m}}(n_j) \to M$ which sends 1 of the j-th copy to β_j. This map is an isomorphism because it induces an isomorphism on socles. By Remark 4.10, we have $\bigoplus_j \frac{\mathcal{D}}{\mathcal{D}\mathfrak{m}}(n_j) \cong \bigoplus_j H^n_{\mathfrak{m}}(R)(-n+n_j)$. Hence $M \cong \bigoplus_j H^n_{\mathfrak{m}}(R)(-n+n_j)$. Since M is Eulerian, so is each $H^n_{\mathfrak{m}}(R)(-n+n_j)$ which implies that $n_j = n$ for each j by Exercise 4.9. This finishes the proof. □

5. Interactions between F-modules and $\mathcal{D}$-modules

We have seen in Proposition 4.3 that each F_R-module is a $\mathcal{D}(R)$-module. It turns out that the theories of F-modules and $\mathcal{D}$-modules are intertwined and their interactions have had many applications. We begin with the following result due to Lyubeznik [Lyu97, 5.7].

Theorem 5.1. *Let R be a regular ring of characteristic p that is a finite R^p-module. Then each F_R-finite module has finite length in the category of $\mathcal{D}(R)$-modules.*

We will see an application of this theorem in a moment.

Exercise 5.2. Let $\phi : R \to S$ be a ring homomorphism between commutative rings such that locally S is a finitely generated free R-module. Then the category of R-modules is equivalent to the one of $\mathrm{End}_R(S)$-modules. The functors are $S \otimes_R -$ and $\mathrm{Hom}_R(S, R) \otimes_{\mathrm{End}_R(S)} -$.

Remark 5.3. We'd like to apply the equivalence in Exercise 5.2 to Frobenius endomorphism.

Let R be a regular ring of characteristic p that is a finite R^p-module. Consider the e-th Frobenius $R \to F^e_* R$ (here we denote the target ring by $F^e_* R$ assuming no confusion will arise). Clearly $\mathrm{End}_R(F^e_* R) \cong \mathrm{End}_{R^{p^e}}(R)$ (keep in mind that this is part of the ring $\mathcal{D}(R)$ according to Remark 4.2). Hence the category of $\mathrm{End}_{R^{p^e}}(R)$-modules is equivalence to the one of R-modules.

One immediate consequence is that the categories of $\mathrm{End}_{R^{p^e}}(R)$-modules are equivalent for all $e \geq 1$. The functor that gives the equivalence between the category of $\mathrm{End}_{R^{p^e}}(R)$-modules and the one of $\mathrm{End}_{R^{p^{e+e'}}}(R)$-modules is $F^{e'}_R(-) = F^{e'}_* R \otimes_R -$. More specifically, let M be a $\mathrm{End}_{R^{p^e}}(R)$-module. The $\mathrm{End}_{R^{p^{e+e'}}}(R)$-module structure on $F^{e'}_R(M)$ can be seen as follows. Since the category of $\mathrm{End}_{R^{p^e}}(R)$-modules is equivalent to the one of R-modules, we know that $M \cong F^e_* R \otimes_R N$ for some R-module N. Then $F^{e'}_R \otimes_R M \cong F^{e+e'}_* R \otimes_R N$. And hence $\mathrm{End}_{R^{p^{e+e'}}}(R)$ acts on $F^{e'}_R(M)$ via its action on $F^{e+e'}_* R \otimes_R N$ which is given by $\alpha \otimes id_N$ for each $\alpha \in \mathrm{End}_{R^{p^{e+e'}}}(R)$.

By our discussion so far, we can see that, when R is a regular ring of characteristic p that is a finite R^p-module, the functor $F^e_R(-)$ is an equivalence of the category of $\mathcal{D}(R)$-modules to itself.

Theorem 5.4. *Let R be a regular ring of characteristic p that is a finite R^p-module. Assume that $\mathscr{M}$ is an F_R-finite module and M is a root of $\mathscr{M}$. Then the image of M in $\mathscr{M}$ generates $\mathscr{M}$ as a $\mathcal{D}(R)$-module.*

Proof. Since M is a root of $\mathscr{M}$, we know that $\beta : M \to F_R(M)$ is injective and hence the direct limit (which is $\mathscr{M}$) of $M \to F_R(M) \to F^2_R(M) \to \cdots$ is a direct union. Thus, we may assume that $M \subset F_R(M)$ and $\mathscr{M} = \bigcup_e F^e_R(M)$.

Let $\mathcal{M}$ be the $\mathcal{D}(R)$-submodule of $\mathscr{M}$ that is generated by M. Since $M \subset F_R(M)$, we have $\mathcal{M} \subset F_R(\mathcal{M})$. If $\mathcal{M}$ were strictly contained in $F_R(\mathcal{M})$ (which is also a $\mathcal{D}(R)$-submodule of $\mathscr{M}$), then we would have a strictly increasing chain of $\mathcal{D}(R)$-submodules of $\mathscr{M}$:

$$\mathcal{M} \subsetneq F_R(\mathcal{M}) \subsetneq F^2_R(\mathcal{M}) \subsetneq \cdots$$

which is a contradiction to Theorem 5.1. Therefore $\mathcal{M} = F_R(\mathcal{M})$. Consequently $\mathscr{M} = \bigcup_e F^e_R(M) \subseteq \bigcup_e F^e_R(\mathcal{M}) = \mathcal{M}$. This finishes the proof. ☐

Corollary 5.5. *Let R be a regular ring of characteristic p that is a finite R^p-module. Assume that $\mathscr{M}$ is an F_R-finite module and $\beta : M \to F_R(M)$ is a generating morphism of $\mathscr{M}$. Then the image of M in $\mathscr{M}$ generates $\mathscr{M}$ as a $\mathcal{D}(R)$-module.*

As an application of the interactions between F-modules and $\mathcal{D}$-modules, we prove the following result [BBL+14, 3.1(1)].

Theorem 5.6. *Let $R = \mathbb{Z}[x_1, \ldots, x_n]$ and $\mathfrak{a}$ be an ideal of R. If a prime number p is a nonzerodivisor on $\mathrm{Ext}^i_R(R/\mathfrak{a}, R)$, then p is also a nonzerodivisor on $H^i_{\mathfrak{a}}(R)$.*

Proof. Let p be a prime integer. The exact sequence $0 \to R \xrightarrow{p} R \to R/pR \to 0$ induces an exact sequence of Ext and an exact sequence of local cohomology modules; they fit into a commutative diagram:

$$\begin{array}{ccccccc}
\operatorname{Ext}^{i-1}_R(R/\mathfrak{a}, R) & \xrightarrow{\psi} & \operatorname{Ext}^{i-1}_R(R/\mathfrak{a}, R/pR) & \longrightarrow & \operatorname{Ext}^{i}_R(R/\mathfrak{a}, R) & \xrightarrow{p} & \operatorname{Ext}^{i}_R(R/\mathfrak{a}, R) \\
\downarrow \beta & & \downarrow \gamma & & \downarrow & & \\
H^{i-1}_{\mathfrak{a}}(R) & \xrightarrow{\varphi} & H^{i-1}_{\mathfrak{a}}(R/pR) & \xrightarrow{\delta} & H^{i}_{\mathfrak{a}}(R) & \xrightarrow{p} & H^{i}_{\mathfrak{a}}(R)
\end{array}$$

The bottom row is a complex of $\mathcal{D}(R)$-modules; in particular, $\varphi\big(H^{i-1}_{\mathfrak{a}}(R)\big)$ is a $\mathcal{D}(R)$-submodule of $H^{i-1}_{\mathfrak{a}}(R/pR)$. As $\varphi\big(H^{i-1}_{\mathfrak{a}}(R)\big)$ is annihilated by p, it has a natural structure as a module over the ring $\mathcal{D}(R)/p\,\mathcal{D}(R)$, which equals $\mathcal{D}(R/pR)$. Similarly,

$$H^{i-1}_{\mathfrak{a}}(R/pR) \xrightarrow{\delta} \operatorname{Im}(\delta) \tag{3}$$

is a map of $\mathcal{D}(R/pR)$-modules.

Suppose p is a nonzerodivisor on $\operatorname{Ext}^i_R(R/\mathfrak{a}, R)$. Then the map ψ is surjective; we need to prove that p is a nonzerodivisor on $H^i_{\mathfrak{a}}(R)$, equivalently, that φ is surjective.

By Corollary 5.5, the image M of γ generates $H^{i-1}_{\mathfrak{a}}(R/pR)$ as a $\mathcal{D}(R/pR)$-module. As ψ is surjective, M is also the image of $\gamma \circ \psi = \varphi \circ \beta$. It follows that

$$M \;\subseteq\; \varphi\big(H^{i-1}_{\mathfrak{a}}(R)\big)\,.$$

But $\varphi\big(H^{i-1}_{\mathfrak{a}}(R)\big)$ is a $\mathcal{D}(R/pR)$-submodule of $H^{i-1}_{\mathfrak{a}}(R/pR)$ that contains M. Hence

$$\varphi\big(H^{i-1}_{\mathfrak{a}}(R)\big) \;=\; H^{i-1}_{\mathfrak{a}}(R/pR)\,,$$

i.e., φ is surjective, as desired. □

Corollary 5.7. *Let $R = \mathbb{Z}[x_1, \ldots, x_n]$ and $\mathfrak{a}$ be an ideal of R. Then there are only finitely many prime numbers p such that $H^j_{\mathfrak{a}}(R)$ contains a nonzero p-torsion element for all j.*

We have seen in Theorem 3.15 that F_R-finite modules have finite length in the category of F_R-module. On the other hand, [Lyu97, 5.7] says that an F_R-finite module with the $\mathcal{D}$-module structure given in Proposition 4.3 also has finite length in the category of $\mathcal{D}$-modules. So, a natural question is how does one compare these two lengths. It turns out that they are the same when the underlying field is algebraically closed.

Theorem 5.8 (Theorem 1.1 in [Bli03])**.** *Let R be a noetherian regular ring that is essentially of finite type over an algebraically closed field of characteristic p. Then the length of an F_R-finite module in the category of F-modules is the same as its length in the category of $\mathcal{D}$-modules.*

Question 5.9. Let $\mathfrak{a}$ be an ideal of R. Can one compute the length of $H^i_{\mathfrak{a}}(R)$ as a $\mathcal{D}$-module?

At the moment, very little is known about Question 5.9; even when $\mathfrak{a}$ is a principal ideal and $i = 1$. We will come back to this question when we discuss non-regular rings in characteristic p.

6. Ring-theoretic properties characterized by Frobenius

Theorem 0.5 exemplifies that the Frobenius endomorphism encodes properties of the ring itself.

Definition 6.1. R is called *F-pure* if $M \to F_*R \otimes_R M$ is injective for all R-modules M.

When $(R, \mathfrak{m})$ is local, R is called *F-injective* if the Frobenius action $H^i_{\mathfrak{m}}(R) \xrightarrow{F} H^i_{\mathfrak{m}}(R)$ is injective all i. More generally when R may not be local, R is called F-injective if $R_{\mathfrak{p}}$ is so for each prime ideal $\mathfrak{p}$.

Exercise 6.2. Let $(R, \mathfrak{m})$ be a noetherian local ring of characteristic p. Let $E(R/\mathfrak{m})$ denote the injective hull of $R/\mathfrak{m}$. Prove that R is F-pure if and only if $E(R/\mathfrak{m}) \to F_*R \otimes_R E(R/\mathfrak{m})$ is injective.

Exercise 6.3. Let $(R, \mathfrak{m})$ be a noetherian local ring of characteristic p. Assume that R is a finite R^p-module. Prove that the following are equivalent

(1) R is F-pure;
(2) for each $e \geq 1$, there is an R^{p^e}-module homomorphism $\varphi : R \to R^{p^e}$ such that the composition $R^{p^e} \hookrightarrow R \xrightarrow{\varphi} R^{p^e}$ is identity on R^{p^e};
(3) for some $e \geq 1$, there is an R^{p^e}-module homomorphism $\varphi : R \to R^{p^e}$ such that the composition $R^{p^e} \hookrightarrow R \xrightarrow{\varphi} R^{p^e}$ is identity on R^{p^e}.

Example 6.4. Let $R = k[[x_1, \dots, x_d]]$ and $f \in \mathfrak{m} = (x_1, \dots, x_d)$ be a formal power series. Since R/fR is a hypersurface and hence Gorenstein, $E((R/fR)/\mathfrak{m}) = H^{d-1}_{\mathfrak{m}}(R/fR)$. So, R/fR is F-pure if and only if the Frobenius action $F : H^{d-1}_{\mathfrak{m}}(R/fR) \to H^{d-1}_{\mathfrak{m}}(R/fR)$ is injective. Consider the following commutative diagram induced by $0 \to R \xrightarrow{\cdot f} R \to R/fR \to 0$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{d-1}_{\mathfrak{m}}(R/fR) & \longrightarrow & H^d_{\mathfrak{m}}(R) & \xrightarrow{\cdot f} & H^d_{\mathfrak{m}}(R) & \longrightarrow & 0 \\ & & \downarrow F & & \downarrow f^{p-1}F & & \downarrow F & & \\ 0 & \longrightarrow & H^{d-1}_{\mathfrak{m}}(R/fR) & \longrightarrow & H^d_{\mathfrak{m}}(R) & \xrightarrow{\cdot f} & H^d_{\mathfrak{m}}(R) & \longrightarrow & 0 \end{array} \tag{4}$$

It is easy to see from this diagram that $F : H^{d-1}_{\mathfrak{m}}(R/fR) \to H^{d-1}_{\mathfrak{m}}(R/fR)$ is injective if the map in the middle $f^{p-1}F$ is injective. From Example 0.3, we can see that this is the case if and only if $f^{p-1} \notin \mathfrak{m}^{[p]}$.

More generally, the quotient ring R/I is F-pure if and only if $(I^{[p]} : I) \nsubseteq \mathfrak{m}^{[p]}$; this is called Fedder's Criterion.

Exercise 6.5. Let $R = \frac{\mathbb{F}_p[[x,y,z]]}{(x^3+y^3+z^3)}$ and consider the element $[\frac{z^2}{xy}] \in H^2_{(x,y,z)}(R)$. Find all prime numbers p such that the image of $[\frac{z^2}{xy}]$ under the Frobenius action on $H^2_{(x,y,z)}(R)$ is 0.

Question 6.6. Assume that f is a polynomial of degree d in d variables $x_1, \dots, x_d$ with integer coefficients and that its Jacobian ideal is primary to $(x_1, \dots, x_d)$. Let f_p denote the image of f under $R := \mathbb{Z}[[x_1, \dots, x_d]] \to R_p := \mathbb{F}_p[[x_1, \dots, x_d]]$. Are there infinitely many prime numbers p such that R_p/f_pR_p is F-pure?

Remark 6.7. When f is a degree 3 homogeneous polynomial that defines an elliptic curve, then by [Sil92, 5.11 on page 145] there are infinitely many prime numbers p such that f_p defines an ordinary elliptic curve and hence R_p/f_pR_p is F-pure.

Exercise 6.8. Assume that $(R, \mathfrak{m})$ is an F-injective local ring. Prove that R is reduced.

Question 6.9. Let f be a regular element of a local ring $(R, \mathfrak{m})$. Assume that R/fR is F-injective. Is it true that R is F-injective?

Exercise 6.10. Let $(R, \mathfrak{m})$ be Cohen-Macaulay and $f \in R$ be a regular element. Assume that R/fR is F-injective. Prove that R is also F-injective.

Question 6.9 is still open in its full generality. For recent developments, see [HMS14].

Definition 6.11. Let R be a noetherian commutative ring of characteristic p. We say that R is strongly F-regular if for every non-zero element $r \in R$ there exists $e \in \mathbb{N}$ and $\phi \in \mathrm{Hom}_{R^{p^e}}(R, R^{p^e})$ such that $\phi(r) = 1$.

Exercise 6.12. Prove that, if R is a strongly F-regular local ring, then R is a domain.

Exercise 6.13. Let $R \subset S$ be a split ring extension (*i.e.* there is an R-module homomorphism $S \to R$ such that $R \to S \to R$ is the identity on R). Assume that S is strongly F-regular. Prove that R is also strongly F-regular.

Question 6.14. Let R be a noetherian commutative ring of characteristic p. An ideal I of R i called *Frobenius closed* if $r^p \in I^{[p]}$ implies $r \in I$ for all elements $r \in R$. Assume now that R is local and consider the following two conditions

(1) R is F-injective;
(2) each parameter ideal I is Frobenius closed.

Is it true that (1) and (2) are equivalent?

Question 6.15. Does the conclusion of Exercise 1.1 still hold when assuming R is strongly F-regular instead of regular?

It turns out that one can use properties of $\mathcal{D}$-modules to characterize strong F-regularity.

Theorem 6.16 (Theorem 2.2(4) in [Smi95])**.** *Let $(R, \mathfrak{m})$ be a noetherian commutative local ring of characteristic p. Assume that R is a finite R^p-module. Then R is strongly F-regular if and only if R is F-pure and a simple $\mathcal{D}$-module.*

Proof. First assume that R is strongly F-regular. Then it is clear that R is F-pure. We will show that $\mathcal{D}(R)r = R$ for each nonzero element $r \in R$. Since R is strongly F-regular, there is an integer $e \geq 1$ and a $\varphi \in \mathrm{Hom}_{R^{p^e}}(R, R^{p^e})$ such that $\varphi(r) = 1$. The composition of φ with $R^{p^e} \hookrightarrow R$, still denoted by φ, is clearly an element of $\mathrm{End}_{R^{p^e}}(R) \subset \mathcal{D}(R)$. Since $\varphi(r) = 1$, we have $\mathcal{D}(R)r = R$.

Conversely, assume that R is F-pure and a simple $\mathcal{D}$-module. Let r be a nonzero element of R. Since R is $\mathcal{D}(R)$-simple, there is $\delta \in \mathcal{D}(R)$ such that $\delta(r) = 1$. By Remark 4.2, δ is R^{p^e}-linear for some $e \geq 1$. By Exercise 6.3, there is a $\varphi \in \mathrm{Hom}_{R^{p^e}}(R, R^{p^e})$ that splits the inclusion $R^{p^e} \hookrightarrow R$. Now $\varphi \circ \delta : R \to R^{p^e}$ is R^{p^e}-linear and $\varphi \circ \delta(r) = 1$. $\square$

Question 6.17. Let R be an integral domain of characteristic p that is a finite R^p-module. Assume that every module-finite ring extension $R \hookrightarrow S$ splits. Is it true that R is $\mathcal{D}(R)$-simple?

The following result due to Blickle [Bli04, 4.10] sheds some light on Question 5.9.

Theorem 6.18. *Let $(R, \mathfrak{m})$ be an F-finite regular local ring and $f \in \mathfrak{m}$ be an element of R. Assume that R/fR is F-injective (or equivalently, F-pure). then $H^1_{(f)}(R)$ is a simple $\mathcal{D}(R)$-module if and only if R/fR is strongly F-regular.*

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