LOCAL COHOMOLOGY AND GROUP ACTIONS

CLAUDIU RAICU

1. Introduction

We consider a polynomial ring $S = \mathbb{C}[x_1, \cdots, x_N]$, and an ideal $I = (f_1, \cdots, f_r)$ in S. The Čech complex

$$0 \longrightarrow S \longrightarrow \bigoplus_i S_{f_i} \longrightarrow \bigoplus_{i<j} S_{f_i f_j} \longrightarrow \cdots \longrightarrow S_{f_1 \cdots f_r} \longrightarrow 0, \tag{1.1}$$

is the tensor product of the complexes $0 \longrightarrow S \longrightarrow S_{f_i} \longrightarrow 0$, defined via the inclusion of S in the localization S_{f_i}, for $i = 1, \cdots, r$. The local cohomology groups $H_I^\bullet(S)$ are the cohomology groups of the Čech complex. They are $\mathcal{D}$-modules, i.e. (left) modules over the Weyl algebra $\mathcal{D} = \mathbb{C}[x_1, \cdots, x_N, \partial_1, \cdots, \partial_N]$ of differential operators with polynomial coefficients ($\partial_i = \frac{\partial}{\partial x_i}$). Moreover, they are holonomic, which in particular implies that they have finite length: they admit a finite filtration (composition series), where the successive quotients (composition factors) are simple $\mathcal{D}$-modules.

Example 1.1. Take $S = \mathbb{C}[x]$, and consider the $\mathcal{D}$-module $S_x = \mathbb{C}[x, 1/x]$. It has a composition series $0 \subset S \subset S_x$, with composition factors S and S_x/S. The latter is the local cohomology group $H^1_{(x)}(S)$.

Exercise 1.2. Verify the assertions in Example 1.1.

Problem 1.3 (Composition series)**.** For a given ideal $I \subset S$, determine which of the local cohomology modules $H_I^\bullet(S)$ are non-zero. For each non-zero local cohomology module, describe its $\mathcal{D}$-module composition factors and their multiplicities.

When G is a group acting compatibly on I and S, the local cohomology groups $H_I^\bullet(S)$ become G-equivariant $\mathcal{D}$-modules, and in particular they are G-representations. In nice situations, when G is a reductive group which is "large enough", the groups $H_I^\bullet(S)$ are admissible representations, i.e. they admit a decomposition into a direct sum (indexed by an usually infinite index set Λ)

$$\bigoplus_{\lambda \in \Lambda} M_\lambda^{\oplus m_\lambda}, \tag{1.2}$$

where M_λ is an irreducible finite dimensional G-representation, and m_λ is the (finite) multiplicity of M_λ.

Problem 1.4 (Characters)**.** When the local cohomology modules $H_I^\bullet(S)$ are admissible G-representations, determine their decomposition (1.2) into irreducible representations.

Example 1.5. If $\mathfrak{m} = (x_1, \cdots, x_N)$ is the ideal of the variables then the only non-vanishing local cohomology module is $H^N_{\mathfrak{m}}(S)$, which is a simple $\mathcal{D}$-module. It is isomorphic to $\mathcal{D}/(x_1, \cdots, x_N)$. It will be useful to use a coordinate free notation: let V be the N-dimensional vector space of linear forms in S, so that $S = \mathrm{Sym}(V)$ is the ring of polynomial functions on the vector space V^*. Note that $\mathrm{GL}(V) = \mathrm{GL}_N(\mathbb{C})$ acts on S by linear changes of coordinates, preserving $\mathfrak{m}$. The local cohomology module $H^N_{\mathfrak{m}}(S)$ is then $\mathrm{GL}(V)$-equivariant.

Date: February 26, 2015.

We have that $H^N_{\mathfrak{m}}(S)$ is an admissible $\mathrm{GL}(V)$-representation, and (writing $S_\lambda V$ for the irreducible $\mathrm{GL}(V)$-representation corresponding to a dominant weight $\lambda = (\lambda_1 \geq \cdots \geq \lambda_N) \in \mathbb{Z}^N$)

$$H^N_{\mathfrak{m}}(S) = \bigwedge^N V^* \otimes \mathrm{Sym}(V^*) = \bigoplus_{r\geq 1} S_{(r,1^{N-1})} V^* = \bigoplus_{r\geq 1} S_{((-1)^{N-1},-r)} V. \tag{1.3}$$

The $\mathrm{GL}(V)$-representations appearing on the right hand side of (1.3) are irreducible. In fact $H^N_{\mathfrak{m}}(S)$ is G-equivariant for any subgroup $G \subset \mathrm{GL}(V)$, but it might not be admissible as a G-representation, or if it is then it might be that the terms in (1.3) are no longer irreducible as G-representations. Nevertheless, even in this case, we regard formula (1.3) as a satisfactory solution to Problem 1.4.

From now on we will denote the module in (1.3) by E, and note that it has many other interpretations:

- E is the injective envelope of the residue field $\mathbb{C} = S/\mathfrak{m}$.
- E is the graded dual of the polynomial ring S.
- E is the unique simple $\mathcal{D}$-module whose support is $\{0\}$.
- E is the Fourier transform of the $\mathcal{D}$-module S.

Exercise 1.6. For every positive integer $r > 0$, compute $\mathrm{Ext}^\bullet_S(S/\mathfrak{m}^r, S)$, both as an S-module and as a $\mathrm{GL}_N(\mathbb{C})$-representation. Prove (1.3) using the direct limit description of local cohomology:

$$H^\bullet_{\mathfrak{m}}(S) = \varinjlim_r \mathrm{Ext}^\bullet_S(S/\mathfrak{m}^r, S).$$

Exercise 1.7. The $\mathcal{D}$-module $S = S_{x_1\cdots x_N} = \mathbb{C}[x_1^{\pm 1}, \cdots, x_N^{\pm 1}]$ is $(\mathbb{C}^*)^N$-equivariant. Determine the composition factors of this $\mathcal{D}$-module, as well as their characters.

The main focus of these notes will be to solve Problems 1.3 and 1.4 for determinantal varieties. For toric varieties, much less is known, and we end this introduction by encouraging you to investigate the following:

Problem 1.8. Let $X \subset \mathbb{P}^N$ be a projective toric variety, embedded by the complete linear system of a torus equivariant line bundle. Let $I_X \subset S = \mathbb{C}[x_0, \cdots, x_N]$ denote the homogeneous ideal of X. Describe the composition factors for the local cohomology modules $H^\bullet_{I_X}(S)$, as well as their multiplicities.

In the case when X is smooth, all of the local cohomology modules in Problem 1.8 (except in degree $\bullet = codim(X)$) will consist of copies of E. The number of such copies is dictated by the singular cohomology of the complement $\mathbb{P}^N \setminus X$ (see [LSW13, Theorem 3.1]). The special case when X is a Veronese variety turns out to have more symmetry than just the one coming from the torus, and the local cohomology modules are admissible representations for a larger group (this case is discussed in [Rai14]). Unfortunately, the torus won't typically be sufficiently large to make the representations $H^\bullet_{I_X}(S)$ admissible, so Problem 1.4 won't apply in this context. It is conceivable that understanding the behavior of direct and inverse images of equivariant $\mathcal{D}$-modules along toric maps (see [dCMM14]) will play an important role in solving Problem 1.8.

2. Equivariant $\mathcal{D}$-modules on $m \times n$ matrices

For a smooth algebraic variety X over $\mathbb{C}$, we let $\mathcal{D}_X$ denote the sheaf of differential operators on X [HTT08, Section 1.1]. A $\mathcal{D}$-module $\mathcal{M}$ on X (or a $\mathcal{D}_X$-module) is a quasi-coherent sheaf $\mathcal{M}$ on X, with a left module action of $\mathcal{D}_X$.

Definition 2.1. Let G be an algebraic group acting on X, and let $\mathcal{M}$ be a $\mathcal{D}_X$-module. Differentiating the action of G on X yields a map $d : Lie(G) \to Der_X$ from the Lie algebra of G to the vector fields on X. The $\mathcal{D}_X$-module operation

$$\mathcal{D}_X \otimes \mathcal{M} \to \mathcal{M}, \tag{2.1}$$

composed with d yields an action of $Lie(G)$ on $\mathcal{M}$. The $\mathcal{D}_X$-module $\mathcal{M}$ is G-equivariant if

(a) $\mathcal{M}$ admits an action of G compatible with (2.1).
(b) The action of $Lie(G)$ on $\mathcal{M}$ obtained by differentiating the action of G on $\mathcal{M}$ coincides with the one induced from $d : Lie(G) \to Der_X$ and (2.1).

Exercise 2.2. Verify that the $\mathcal{D}$-module in Exercise 1.7 is indeed $(\mathbb{C}^*)^N$-equivariant.

The following theorem gives a classification of the simple equivariant $\mathcal{D}$-modules, for a group action with finitely many orbits (see [HTT08, Section 11.6]):

Theorem 2.3. *Let G be an algebraic group acting on a smooth algebraic variety X with finitely many orbits. There is a one-to-one correspondence between:*

(a) *Simple G-equivariant $\mathcal{D}$-modules on X.*
(b) *Pairs (O, L) where O is a G-orbit, and L is an irreducible representation of the component group of the isotropy group of O.*

Here by the isotropy group of O we mean the stabilizer of any element in O (they are all isomorphic). For an algebraic group H, we denote by H_0 the connected component of the identity, which is a normal subgroup of H. The quotient H/H_0 is the component group of H.

Exercise 2.4. Classify all the simple $(\mathbb{C}^*)^N$-equivariant $\mathcal{D}$-modules on $X = \mathbb{A}^N$. Show that they are precisely the composition factors of the $\mathcal{D}$-module $S_{x_1 \cdots x_N}$ in Exercise 1.7. Show that they all appear as local cohomology modules $H^\bullet_I(S)$ for appropriate ideals I.

Example 2.5. Let $d > 0$ be a positive integer and let $G = \mathbb{C}^*$ act on $X = \mathbb{A}^1$ via, $t \cdot a = t^d a$ for $t \in \mathbb{C}^*$ and $a \in \mathbb{A}^1$. If we let $S = \mathbb{C}[x]$ denote the coordinate ring of $\mathbb{A}^1$, then $\mathbb{C}^*$ acts on S via $t \cdot x = t^{-d}x$. There are two orbits for this action, namely $O_0 = \{0\}$ and $O_1 = \mathbb{A}^1 \setminus \{0\}$. The isotropy group of O_0 is $\mathbb{C}^*$, which is connected. We get one simple $\mathcal{D}$-module corresponding to O_0, namely $S_x/S(= E)$. The isotropy group of O_1 has d connected components, and its component group is cyclic of order d. The corresponding $\mathcal{D}$-modules are S (which corresponds to the trivial representation), and $M_i = x^{-i/d}S_x$, for $i = 1, \cdots, d-1$. The action of $\mathbb{C}^*$ on M_i is given by $t \cdot x^{-i/d+n} = t^{i-dn}x^{-i/d+n}$.

Exercise 2.6. Show that the $\mathcal{D}$-modules in Example 2.5 are simple, $\mathbb{C}^*$-equivariant, and pairwise non-isomorphic.

We now focus on the case when X is the vector space of $m \times n$ matrices, $m \geq n$, and $G = \mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ acts via row and column operations. The orbit decomposition is $X = O_0 \cup O_1 \cup \cdots \cup O_n$, where O_i is the orbit of matrices of rank i. Its isotropy group is connected, so each O_i gives rise to precisely one simple G-equivariant $\mathcal{D}$-module, which we denote by D_i. One has $D_0 = E$, $D_n = S$, but the other D_i's are more mysterious. We start by describing their characters. We write $\mathbb{Z}^n_{dom}$ for the collection of dominant weights $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n) \in \mathbb{Z}^n$. Given $\lambda \in \mathbb{Z}^n_{dom}$ and $0 \leq s \leq n$, we define

$$\lambda(s) = (\lambda_1, \cdots, \lambda_s, \underbrace{s-n, \cdots, s-n}_{m-n}, \lambda_{s+1} + (m-n), \cdots, \lambda_n + (m-n)) \in \mathbb{Z}^m. \tag{2.2}$$

Theorem 2.7 ([Rai15],[RW14a])**.** *The decomposition into irreducible* $\mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$*-representations of the $\mathcal{D}$-module D_s is given by:*

$$D_s = \bigoplus_{\substack{\lambda \in \mathbb{Z}^n_{dom} \\ \lambda_s \geq s-n \\ \lambda_{s+1} \leq s-m}} S_{\lambda(s)}\mathbb{C}^m \otimes S_\lambda \mathbb{C}^n. \tag{2.3}$$

We obtain the following description of the $\mathcal{D}$-modules D_s in the case when $m = n$.

Corollary 2.8. *Assume that $m = n$ and write $det = det(x_{ij})$ for the determinant of the generic $n \times n$ matrix. For $s = 0, \cdots, n$ we let $F_s = \langle det^{s-n} \rangle_{\mathcal{D}}$ denote the $\mathcal{D}$-module generated by det^{s-n}. The $\mathcal{D}$-module S_{det} has a composition series $0 \subset S = F_n \subset F_{n-1} \subset \cdots \subset F_0 = S_{det}$, where $F_s/F_{s+1} \simeq D_s$.*

Exercise 2.9. Prove Corollary 2.8.

3. Bernstein-Sato polynomials

Let $S = \mathbb{C}[x_1, \cdots, x_N]$, and let $\mathcal{D}$ denote the associated Weyl algebra. Given a non-zero element $f \in S$, the Bernstein–Sato polynomial of f, denoted $b_f = b_f(s)$, is the monic polynomial of minimal degree for which there exists a differential operator $P \in \mathcal{D}[s]$ such that

$$P \cdot f^{s+1} = b_f(s) \cdot f^s. \tag{3.1}$$

$b_f(s)$ should be interpreted as a measure of the singularity of the affine variety $Z(f)$ defined by the vanishing of f. $Z(f)$ is smooth precisely when $b_f(s)$ is a linear form.

Exercise 3.1. Show that for $f = x_1^2 + \cdots + x_N^2$,

$$b_f(s) = (s+1) \cdot (s + N/2).$$

We are not aware of any counterexample to the following:

Conjecture 3.2. *For a rational number $s \in \mathbb{Q}$, f^s and f^{s+1} generate distinct $\mathcal{D}$-modules if and only if s is a root of b_f.*

One implication of this conjecture is clear: if s is not a root of b_f then $b_f(s) \neq 0$, and (3.1) shows that f^s is contained in the $\mathcal{D}$-module generated by f^{s+1}. The case of interest for us is when $S = \mathbb{C}[x_{ij} : i, j = \overline{1, n}]$ is the ring of polynomial functions on the vector space of $n \times n$ complex matrices, and $f = \det = \det(x_{ij})$. As you will prove shortly, the Bernstein-Sato polynomial of the generic determinant is given by:

$$b_{\det}(s) = (s+1) \cdot (s+2) \cdots (s+n). \tag{3.2}$$

Exercise 3.3. In the case $n = 2$, prove via a direct calculation that $\det^{-2}$ is not contained in the $\mathcal{D}$-module generated by $\det^{-1}$.

Exercise 3.4. Use Corollary 2.8 in order to prove Conjecture 3.2 in the special case when f is the determinant of the generic $n \times n$ matrix.

Cayley's identity (see [CSS13, Sec. 2.6] for some historical remarks)

$$\det(\partial_{ij}) \cdot \det^{s+1} = (s+1) \cdot (s+2) \cdots (s+n) \cdot \det^s \tag{3.3}$$

provides an upper bound for the Bernstein-Sato polynomial of the generic determinant.

Exercise 3.5. Prove Cayley's identity (3.3) in the case when $n = 2$.

Exercise 3.6. Prove (3.2) using Cayley's identity and Corollary 2.8.

The only other approach for computing the Bernstein-Sato polynomial of the determinant that we are aware of is based on the theory of prehomogeneous vector spaces [Kim03].

Exercise 3.7. Determine $b_f(s)$ for $f = x_1^{a_1} \cdots x_r^{a_r}$, where $a_1, \cdots, a_r$ are positive integers.

Consider now the generic $m \times n$ matrix (x_{ij}), and let I_p denote the ideal generated by its $p \times p$ minors. The Bernstein-Sato polynomial $b_I(s)$ can be defined for an arbitrary ideal I in a polynomial ring, and even more generally in the non-affine setting [BMS06]. The following Conjecture appears, in the case $m = n$, in [Bud13]:

Conjecture 3.8. *The Bernstein–Sato polynomial of the ideal of $p \times p$ minors of the generic $m \times n$ matrix is*

$$b_{I_p}(s) = \prod_{i=0}^{p-1} \left(s + \frac{(m-i)\cdot(n-i)}{p-i} \right).$$

4. Local cohomology with support in determinantal ideals

For positive integers $m \geq n$, we consider the ring $S = \mathrm{Sym}(\mathbb{C}^m \otimes \mathbb{C}^n)(= \mathbb{C}[x_{ij}])$ of polynomial functions on the vector space of $m \times n$ complex matrices. The ring S admits an action of the group $\mathrm{GL} = \mathrm{GL}_m(\mathbb{C}) \otimes \mathrm{GL}_n(\mathbb{C})$. For $p = 1, \cdots, n$, we let I_p denote the ideal of $p \times p$ minors of the generic matrix (x_{ij}). In this section we solve Problems 1.3 and 1.4 for the ideals $I_p \subset S$. To state the results, we need to introduce some notation.

We let $\Gamma(\mathrm{GL})$ denote the Grothendieck group of admissible GL-representations. $\Gamma(\mathrm{GL})$ is a direct product of copies of $\mathbb{Z}$, indexed by $\{S_\lambda \mathbb{C}^m \otimes S_\mu \mathbb{C}^n : \lambda \in \mathbb{Z}^m_{dom}, \mu \in \mathbb{Z}^n_{dom}\}$, the set of irreducible finite dimensional representations of GL. For $0 \leq s \leq n$, we define $h_s \in \Gamma(\mathrm{GL})$ via

$$h_s = \sum_{\substack{\lambda \in \mathbb{Z}^n_{dom} \\ \lambda_s \geq s-n \\ \lambda_{s+1} \leq s-m}} S_{\lambda(s)} F \otimes S_\lambda G. \tag{4.1}$$

If we denote by $[M]_{\Gamma(\mathrm{GL})}$ the class in $\Gamma(\mathrm{GL})$ of an admissible representation M, then $h_s = [D_s]_{\Gamma(\mathrm{GL})}$ (see (2.3)).

We write $p(a, b; c)$ for the number of partitions of c contained in an $a \times b$ rectangle, and define the Gauss polynomial (or Gaussian binomial coefficient) $\binom{a+b}{b}_w$ to be the generating function for the sequence $p(a, b; c)_{c \geq 0}$:

$$\binom{a+b}{a}_w = \sum_{c \geq 0} p(a, b; c) \cdot w^c = \sum_{b \geq t_1 \geq t_2 \geq \cdots \geq t_a \geq 0} w^{t_1 + \cdots + t_a} = \frac{(1-w^{a+b})(1-w^{a+b-1})\cdots(1-w^{a+1})}{(1-w^b)(1-w^{b-1})\cdots(1-w)}. \tag{4.2}$$

The polynomial $\binom{a+b}{a}_{w^2}$ can also be interpreted as the Poincaré polynomial of the Grassmannian of a planes in $(a+b)$-dimensional space.

Exercise 4.1. Verify the 2nd and 3rd equalities in (4.2).

Theorem 4.2 ([RW14a],[RWW14]). *For each $p = 1, \cdots, n$, the local cohomology groups $H^\bullet_{I_p}(S)$ are admissible GL-representations. If we define formal polynomials $H_p^{\mathrm{GL}}(w)$ with coefficients in $\Gamma(\mathrm{GL})$ via*

$$H_p^{\mathrm{GL}}(w) = \sum_{j=0}^{mn} [H^j_{I_p}(S)]_{\Gamma(\mathrm{GL})} \cdot w^j,$$

then using the notation (4.1) we have

$$H_p^{\mathrm{GL}}(w) = \sum_{s=0}^{p-1} h_s \cdot w^{(n-p+1)^2 + (n-s)\cdot(m-n)} \cdot \binom{n-s-1}{p-s-1}_{w^2}.$$

Theorem 4.2 provides a complete answer to Problem 1.4, so we now have to address Problem 1.3. We let $\Gamma(X,\mathrm{GL})$ denote the Grothendieck group of GL-equivariant holonomic $\mathcal{D}_X$-modules, where X is the vector space of $m \times n$ matrices. It is the free abelian group on the set $\{D_0, D_1, \cdots, D_n\}$ of simple equivariant $\mathcal{D}$-modules described in Section 2. We get

Corollary 4.3. *If we define formal polynomials $H_p^{\mathcal{D}}(w)$ with coefficients in $\Gamma(X,\mathrm{GL})$ via*

$$H_p^{\mathcal{D}}(w) = \sum_{j=0}^{mn} [H^j_{I_p}(S)]_{\Gamma(X,\mathrm{GL})} \cdot w^j,$$

then

$$H_p^{\mathcal{D}}(w) = \sum_{s=0}^{p-1} [D_s]_{\Gamma(X,\mathrm{GL})} \cdot w^{(n-p+1)^2+(n-s)\cdot(m-n)} \cdot \binom{n-s-1}{p-s-1}_{w^2}.$$

Exercise 4.4. Prove Corollary 4.3, and use it to describe the $\mathcal{D}$-modules $D_0, \cdots, D_{n-1}$ as local cohomology modules in the case when $m > n$ (compare with Corollary 2.8).

Theorem 4.2 implies that the maximal cohomological index for which $H^{\bullet}_{I_p}(S)$ is non-zero (the cohomological dimension of the ideal I_p) is obtained for $s = 0$ and is equal to

$$(n-p+1)^2 + n\cdot(m-n) + (p-1)\cdot(n-p) = m\cdot n - p^2 + 1.$$

This was first observed in [BS90]. Using the fact that once we invert one of the entries of a generic $m \times n$ matrix, I_p becomes I_{p-1} for a generic $(m-1)\times(n-1)$ matrix, it follows easily from the above that

$$H^j_{I_p}(S) \neq 0 \text{ for } j = (m-s)\cdot(n-s) - (p-s)^2 + 1,\ s = 0,1,\cdots,p-1. \tag{4.3}$$

For maximal minors ($p = n$) this non-vanishing result is sharp, and this was first explained in [Wit12]. The next result, which is a direct consequence of Theorem 4.2, says that *many more* of the local cohomology modules $H^j_{I_p}(S)$ are non-zero when $p < n$, namely:

Corollary 4.5. *If $p \leq n \leq m$ then $H^j_{I_p}(S) \neq 0$ precisely when*

$$j = (n-p+1)^2 + (n-s)\cdot(m-n) + 2\cdot k,\ \text{for } 0 \leq s \leq p-1 \text{ and } 0 \leq k \leq (p-s-1)\cdot(n-p).$$

The non-vanishing statement (4.3) is obtained for $k = (p-s-1)\cdot(n-p)$.

Exercise 4.6. Prove Corollary 4.5.

The conclusion of Corollary 4.5 contrasts with the positive characteristic situation where the only non-vanishing local cohomology module appears in degree $j = (m-p+1)\cdot(n-p+1)$ (see [HE71, Cor. 4] or [BV88, Cor. 5.18] where it is shown that I_p is perfect, and [PS73, Prop. 4.1] where a local cohomology vanishing result for perfect ideals in positive characteristic is proved; see also Wenliang's lecture). For determinantal ideals over arbitrary rings one can't expect such explicit results as Theorem 4.2, but see [LSW13] and the references therein.

5. Cohen-Macaulayness of modules of covariants

One of the motivations behind the investigation of characters of local cohomology modules is an attempt to characterize the Cohen-Macaulay property for modules of covariants. This problem has a long history, originating in the work of Stanley [Sta82] on solution sets of linear Diophantine equations (see [VdB95] for a survey, and [VdB91]). When G is a reductive group and W is a finite dimensional G-representation over the complex numbers, a celebrated theorem of Hochster and Roberts [HR74] asserts that the ring of invariants

S^G, with respect to the natural action of G on the polynomial ring $S = \mathrm{Sym}(W)$, is Cohen-Macaulay. If U is another finite dimensional G-representation, the associated module of covariants is defined as $(S \otimes U)^G$, and is a finitely generated S^G-module.

Problem 5.1. For a reductive group G, and a finite dimensional complex G-representation W, consider $S = \mathrm{Sym}(W)$ with the induced G-action. Classify the irreducible representations U for which the module of covariants $(S \otimes U)^G$ is Cohen-Macaulay.

Exercise 5.2. Let $\mathfrak{m}$ denote the maximal homogeneous ideal of the ring of invariants S^G, and let $I = \mathfrak{m}S$ denote its extension to an ideal of S. Show that

$$H^j_{\mathfrak{m}}((S \otimes U)^G) = (H^j_I(S) \otimes U)^G.$$

The module of covariants $(S \otimes U)^G$ is Cohen-Macaulay if and only if the local cohomology groups $H^j_{\mathfrak{m}}((S \otimes U)^G)$ vanish for $j < \dim(S^G)$. Therefore Exercise 5.2 shows that in order to prove the (non-)vanishing of local cohomology for a module of covariants, it is sufficient to understand when U^* appears as a subrepresentation of $H^j_I(S)$.

When the representation W is "small", the ring of invariants S^G is trivial and all the modules of covariants are Cohen-Macaulay. For "large" W however, it is quite rare that $(S \otimes U)^G$ is Cohen-Macaulay. The following result illustrates this in a special situation.

Theorem 5.3 ([RWW14]). *Let $m > n$ be positive integers, and let $G = \mathrm{SL}_n(\mathbb{C})$ be the special linear group, and $W = (\mathbb{C}^n)^{\oplus m}$ be a direct sum of m copies of the standard representation. If $S = \mathrm{Sym}(W)$, and $U = S_\mu \mathbb{C}^n$ is the irreducible G-representation associated to the partition $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0)$ then $(S \otimes U)^G$ is Cohen-Macaulay if and only if $\mu_s - \mu_{s+1} < m - n$ for all $s = 1, \cdots, n-1$.*

In the case when $m = n + 1$, the Theorem on Covariants asserts that the only Cohen-Macaulay modules of covariants are direct sums of copies of S^G, which is remarked at the end of [Bri93]. The case $n = 3$ of the theorem is explained in [VdB99], while the case $n = 2$ for an arbitrary G-representation W is treated in [VdB94].

Exercise 5.4. Show that the ring of invariants S^G for $G = \mathrm{SL}_n(\mathbb{C})$ and $S = \mathrm{Sym}(\mathbb{C}^m \otimes \mathbb{C}^n)$ is the $\mathbb{C}$-algebra generated by the $n \times n$ minors of the generic $m \times n$ matrix.

Exercise 5.5. With the notation from Theorem 5.3, which modules of covariants are Cohen-Macaulay in the case when $m \leq n$?

Exercise 5.6. Use the special case $p = n$ of Theorem 4.2 and Exercises 5.2 and 5.4 in order to prove Theorem 5.3.

6. Syzygies of thickenings of determinantal ideals

For positive integers $m \geq n$, let $S = \mathrm{Sym}(\mathbb{C}^m \otimes \mathbb{C}^n)$ be the ring of polynomial functions on the vector space of $m \times n$ complex matrices, and let $\mathrm{GL} = \mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$. The ring S decomposes into irreducible GL-representations according to Cauchy's formula:

$$S = \bigoplus_{\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0)} S_\lambda \mathbb{C}^m \otimes S_\lambda \mathbb{C}^n.$$

For each λ, we let I_λ denote the ideal in S generated by the irreducible representation $S_\lambda \mathbb{C}^m \otimes S_\lambda \mathbb{C}^n$. Every ideal $I \subset S$ which is preserved by the GL-action is a sum of ideals I_λ: such ideals I have been classified and their geometry has been studied by De Concini, Eisenbud and Procesi in the 80s [dCEP80]. Nevertheless, their syzygies are still mysterious, and in particular the following problem remains unsolved:

Problem 6.1. Describe the syzygies of the ideals I_λ, together with their structure as GL-representations.

Exercise 6.2. Classify the GL-invariant ideals in the case when $n = 1$, and compute their syzygies (see Exercise 8.2).

In what follows we explain the solution to Problem 6.1 in the case when λ is a rectangular partition, which means that there exist positive integers a, b such that $\lambda_1 = \cdots = \lambda_a = b$ and $\lambda_i = 0$ for $i > a$ (alternatively, the Young diagram associated to λ is the $a \times b$ rectangle). In this case we write $\lambda = a \times b$ and $I_\lambda = I_{a\times b}$. One can think of $I_{a\times b}$ as the smallest GL-equivariant ideal which contains the b-th powers of the $a \times a$ minors of the generic matrix of indeterminates (x_{ij}). What distinguishes the ideals $I_{a\times b}$ among all the I_λ's is that they define a scheme without embedded components, so from a geometric point of view they form the simplest class of GL-equivariant ideals after the reduced (and prime) ideals of minors. Examples of ideals $I_{a\times b}$ include:

- $I_{a\times 1} = I_a$, the ideal generated by the $a \times a$ minors of (x_{ij}).
- $I_{n\times b} = I_n^b$, the b-th power of the ideal I_n of maximal minors of (x_{ij}).
- $I_{1\times b}$, the ideal of $b\times b$ permanents of (x_{ij}): here by a $b\times b$ permanent of (x_{ij}) we mean the permanent of a $b \times b$ matrix obtained by selecting b rows and b columns of (x_{ij}), not necessarily distinct; for instance, when $m = n = 2$ we have:

$$I_{1\times 2} = (x_{11}^2, x_{12}^2, x_{21}^2, x_{22}^2, x_{11}x_{12}, x_{11}x_{21}, x_{12}x_{22}, x_{21}x_{22}, x_{11}x_{22} + x_{12}x_{21}). \tag{6.1}$$

Exercise 6.3. Verify that the ideal $I_{1\times 2}$ in (6.1) is preserved by the GL-action.

To state the main result, we'll need some notation. Write $\mathcal{R} = Rep_{\mathrm{GL}}$ for the representation ring of the group GL, and for a given GL-representation M, let $[M]_{\mathcal{R}}$ denote its class in $\mathcal{R}$. Let

$$B_{i,j}(I_{a\times b}) = \mathrm{Tor}_i^S(I_{a\times b}, \mathbb{C})_j \tag{6.2}$$

denote the vector space of i-syzygies of degree j of $I_{a\times b}$. We encode the syzygies of $I_{a\times b}$ into the equivariant Betti polynomial

$$B_{a\times b}(z, w) = \sum_{i,j\in\mathbb{Z}} [B_{i,j}(I_{a\times b})]_{\mathcal{R}} \cdot w^i \cdot z^j \in \mathcal{R}[z, w], \tag{6.3}$$

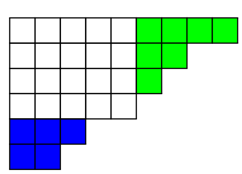
so the variable z keeps track of the internal degree, while w keeps track of the homological degree.

If r, s are positive integers, α is a partition with at most r parts ($\alpha_i = 0$ for $i > r$) and β is a partition with parts of size at most s ($\beta_1 \leq s$), we construct the partition

$$\lambda(r, s; \alpha, \beta) = (r + \alpha_1, \cdots, r + \alpha_r, \beta_1, \beta_2, \cdots). \tag{6.4}$$

This is easiest to visualize in terms of Young diagrams: one starts with an $r \times s$ rectangle, and attach α to the right and β to the bottom of the rectangle. If $r = 4$, $s = 5$, $\alpha = (4, 2, 1)$, $\beta = (3, 2)$, then

$$\lambda(r, s; \alpha, \beta) = \tag{6.5}$$



We write μ' for the conjugate partition to μ (obtained by transposing the Young diagram of μ) and consider the polynomials $h_{r\times s} \in \mathcal{R}[z, w]$ given by

$$h_{r\times s}(z, w) = \sum_{\alpha,\beta} S_{\lambda(r,s;\alpha,\beta)}\mathbb{C}^m \otimes S_{\lambda(r,s;\beta',\alpha')}\mathbb{C}^n \cdot z^{r\cdot s+|\alpha|+|\beta|} \cdot w^{|\alpha|+|\beta|}, \tag{6.6}$$

where the sum is taken over partitions α, β such that α is contained in the $\min(r,s) \times (n-r)$ rectangle ($\alpha_1 \leq n-r$, $\alpha'_1 \leq \min(r,s)$) and β is contained in the $(m-r) \times \min(r,s)$ rectangle ($\beta_1 \leq \min(r,s)$ and $\beta'_1 \leq m-r$). With the notation (4.2) for Gauss polynomials, we have:

Theorem 6.4 ([RW14b]). *The equivariant Betti polynomial of $I_{a\times b}$ is*

$$B_{a\times b}(z,w) = \sum_{q=0}^{n-a} h_{(a+q)\times(b+q)} \cdot w^{q^2+2q} \cdot \binom{q+\min(a,b)-1}{q}_{w^2}$$

When $b = 1$, this recovers the result of Lascoux on syzygies of determinantal varieties [Las78]. When $a = n$, we obtain the syzygies of the powers of the ideals of maximal minors, as originally computed by Akin–Buchsbaum–Weyman [ABW81].

Exercise 6.5. Figure out what Theorem 6.4 says in the case when $n = 1$, and compare to your result from Exercise 6.2.

Exercise 6.6. Determine the Betti table of the ideal $I_{1\times 2}$ from (6.1) in two ways: (a) using Theorem 6.4; (b) using Macaulay [GS].

7. Bott's Theorem for Grassmannians

In order to solve the exercises in the subsequent sections, it will be important to be able to compute sheaf cohomology for certain GL-equivariant locally free sheaves on Grassmannians. This is the content of Bott's theorem which we explain in this section.

Consider a finite dimensional complex vector space W of dimension N, and a non-negative integer $k \leq N$. Consider the Grassmannian $\mathbb{G} = \mathbb{G}(k, W)$ of k–dimensional quotients of W. The dimension of $\mathbb{G}$ is $d_{\mathbb{G}} = k(N-k)$. On $\mathbb{G}$ we have the tautological exact sequence

$$0 \longrightarrow \mathcal{R} \longrightarrow W \otimes \mathcal{O}_{\mathbb{G}} \longrightarrow \mathcal{Q} \longrightarrow 0, \tag{7.1}$$

where $\mathcal{R}$ (resp. $\mathcal{Q}$) denotes the universal sub–bundle (resp. quotient bundle). We have $\operatorname{rank}(\mathcal{Q}) = k$ and $\operatorname{rank}(\mathcal{R}) = N-k$. The sheaf $\Omega^i_{\mathbb{G}}$ of i-differential forms on $\mathbb{G}$ is given by $\Omega^i_{\mathbb{G}} = \bigwedge^i(\mathcal{R} \otimes \mathcal{Q}^*)$. When $\mathcal{E}$ is a locally free sheaf with $\operatorname{rank}(\mathcal{E}) = e$, we write $\det(\mathcal{E}) = \bigwedge^e \mathcal{E}$ for its top exterior power. It will be useful to note that

$$S_{(\lambda_1,\cdots,\lambda_e)}\mathcal{E}^* = S_{(-\lambda_e,\cdots,-\lambda_1)}\mathcal{E}, \text{ and } S_\lambda\mathcal{E} \otimes \det(\mathcal{E}) = S_{\lambda+(1^e)}\mathcal{E},$$

and also that the canonical line bundle on $\mathbb{G}$ is

$$\omega_{\mathbb{G}} = \det(\mathcal{R} \otimes \mathcal{Q}^*) = \det(\mathcal{R})^k \otimes \det(\mathcal{Q})^{k-N}.$$

Consider two dominant weights $\alpha = (\alpha_1, \cdots, \alpha_k)$ and $\beta = (\beta_1, \cdots, \beta_{N-k})$, and their concatenation $\gamma = (\gamma_1, \cdots, \gamma_N)$. Let $\delta = (N-1, \cdots, 0)$, and consider $\gamma + \delta = (\gamma_1 + N - 1, \gamma_2 + N - 2, \cdots, \gamma_N)$. We write $\operatorname{sort}(\gamma+\delta)$ for the sequence obtained by arranging the entries of $\gamma + \delta$ in non-increasing order, and define

$$\tilde{\gamma} = \operatorname{sort}(\gamma + \delta) - \delta. \tag{7.2}$$

Example 7.1. If $k = 2$, $N = 5$, $\alpha = (3,1)$ and $\beta = (4,4,2)$ then $\gamma = (3,1,4,4,2)$, $\delta = (4,3,2,1,0)$, $\gamma+\delta = (7,4,6,5,2)$, $\operatorname{sort}(\gamma+\delta) = (7,6,5,4,2)$, and $\tilde{\gamma} = (3,3,3,3,2)$.

Theorem 7.2 (Bott). *With the above notation, if $\gamma+\delta$ has repeated entries, then*

$$H^t(\mathbb{G}, S_\alpha\mathcal{Q} \otimes S_\beta\mathcal{R}) = 0 \text{ for all } t \geq 0.$$

Otherwise, writing l for the number of pairs (x,y) with $1 \leq x < y \leq N$ and $\gamma_x - x < \gamma_y - y$ (which is the number of inversions of the permutation that realizes the sorting of $\gamma + \delta$), we have

$$H^l(\mathbb{G}, S_\alpha\mathcal{Q} \otimes S_\beta\mathcal{R}) = S_{\tilde{\gamma}}W,$$

and $H^t(\mathbb{G}, S_\alpha\mathcal{Q} \otimes S_\beta\mathcal{R}) = 0$ for $t \neq l$.

Exercise 7.3. Compute the cohomology groups $H^\bullet(\mathbb{G}, S_{(3,1)}\mathcal{Q} \otimes S_{(4,4,2)}\mathcal{R})$, where $\mathbb{G} = \mathbb{G}(2, \mathbb{C}^5)$ as in Example 7.1.

Exercise 7.4. Prove Serre duality for locally free sheaves of the form $S_\alpha\mathcal{Q} \otimes S_\beta\mathcal{R}$:

$$H^j(\mathbb{G}, S_\alpha\mathcal{Q} \otimes S_\beta\mathcal{R}) = H^{d_{\mathbb{G}}-j}(\mathbb{G}, \omega_{\mathbb{G}} \otimes S_\alpha\mathcal{Q}^* \otimes S_\beta\mathcal{R}^*)^*.$$

8. The geometric technique for computing syzygies and Ext modules

Let X be a smooth projective complex algebraic variety and denote its dimension by d_X. Consider a finite dimensional vector space U, and a short exact sequence

$$0 \longrightarrow \xi \longrightarrow U \otimes \mathcal{O}_X \longrightarrow \eta \longrightarrow 0, \tag{8.1}$$

where ξ, η are locally free sheaves on X. We think of U^* as an affine space, and of U as linear forms on U^*. We let $Y = \mathrm{Tot}_X(\eta^*)$ denote the total space of the bundle η^*, and define a morphism $\pi : Y \to U^*$ via the commutative diagram

$$\begin{array}{ccc} Y = \mathrm{Tot}_X(\eta^*) & \hookrightarrow & U^* \times X \\ & \overset{\pi}{\searrow} & \downarrow \\ & & U^* \end{array} \tag{8.2}$$

where the top map is the inclusion of η^* into the trivial bundle U^*, and the vertical map is the projection onto the U^* factor. We let $S = \mathrm{Sym}(U)$ and $\mathcal{S} = \mathrm{Sym}_{\mathcal{O}_X}(\eta)$.

Given a locally free coherent sheaf $\mathcal{V}$ on X, we define $\mathcal{M}(\mathcal{V})$ by

$$\mathcal{M}(\mathcal{V}) = \mathcal{V} \otimes \mathcal{S}. \tag{8.3}$$

We think of $\mathcal{M}(\mathcal{V})$ as a free $\mathcal{S}$-module. The natural surjective map $S \otimes \mathcal{O}_X \to \mathcal{S}$ yields an S-module structure on

$$M(\mathcal{V}) = H^0(X, \mathcal{M}(\mathcal{V})). \tag{8.4}$$

The Koszul complex on ξ yields resolution $\mathcal{K}_\bullet(\mathcal{V})$ of $\mathcal{M}(\mathcal{V})$ by free $S \otimes \mathcal{O}_X$-modules with

$$\mathcal{K}_i(\mathcal{V}) = \mathcal{V} \otimes \bigwedge^i \xi \otimes S \tag{8.5}$$

When $\mathcal{M}(\mathcal{V})$ has vanishing higher cohomology, pushing forward the complex $\mathcal{K}_\bullet(\mathcal{V})$ yields a resolution of $M(\mathcal{V})$:

Theorem 8.1 ([Wey03, Section 5.1])**.** *With notation as above, assume that $H^j(X, \mathcal{M}(\mathcal{V})) = 0$ for $j > 0$. The syzygies of the S-module $M(\mathcal{V})$ can be computed via*

$$\mathrm{Tor}_i^S(M(\mathcal{V}), \mathbb{C})_{i+j} = H^j\left(X, \mathcal{V} \otimes \bigwedge^{i+j} \xi\right).$$

Exercise 8.2. Compute the syzygies of the powers of the maximal homogeneous ideal $\mathfrak{m} = (x_1, \cdots, x_N)$ in a polynomial ring $S = \mathbb{C}[x_1, \cdots, x_N]$.

Exercise 8.3. Compute the syzygies of the powers of the ideal of $n \times n$ minors of the generic $m \times n$ matrix. Compare your answer with Theorem 6.4.

We define the quasi-coherent sheaf $\mathcal{S}^\vee$ on X (the graded dual of $\mathcal{S}$) by

$$\mathcal{S}^\vee = \det(\eta^*) \otimes \mathrm{Sym}_{\mathcal{O}_X}(\eta^*). \tag{8.6}$$

The following theorem yields a useful method for computing Ext modules:

Theorem 8.4 ([RWW14, Theorem 3.1])**.** *We assume that $H^j(X, \mathcal{M}(\mathcal{V})) = 0$ for $j > 0$, denote by k the rank of the locally free sheaf ξ, and define*

$$\mathcal{M}^\vee(\mathcal{V}) = \mathcal{V} \otimes \mathcal{S}^\vee. \tag{8.7}$$

We have

$$\mathrm{Ext}^j_S(M(\mathcal{V}), S) = H^{k-j}(X, \mathcal{M}^\vee(\mathcal{V}))^* \otimes det(U^*).$$

Exercise 8.5. Compute $\mathrm{Ext}^\bullet_S(\mathfrak{m}^r, S)$ for $\mathfrak{m} = (x_1, \cdots, x_N)$ and compare with Exercise 1.6.

Exercise 8.6. Compute $\mathrm{Ext}^\bullet_S(I^r, S)$ when I is the ideal of $n \times n$ minors of the generic $m \times n$ matrix (see Section 9.1 if you get stuck).

9. Some special cases of the main theorems

In this section we give a flavor of the techniques involved in proving Theorems 2.7 and 4.2 by analyzing some special cases. We compute the character of the $\mathcal{D}$-module D_{n-1} (see (2.3)), and determine the characters of the local cohomology modules with support in the ideal of $n \times n$ minors of the generic $m \times n$ matrix.

9.1. **Proof of Theorem 4.2 in the case when $p = n$.** We need to compute the local cohomology modules $H^\bullet_I(S)$, where I is the ideal of $n \times n$ minors of the generic $m \times n$ matrix. We will be using the expression of the local cohomology modules as a direct limit of Ext modules:

$$H^\bullet_I(S) = \varinjlim_d \mathrm{Ext}^\bullet_S(S/I^d, S). \tag{9.1}$$

Exercise 9.1. Verify Theorem 4.2 in the case $m = n = p$.

Exercise 9.2. Show that when $m > n$ we have $\mathrm{Ext}^0_S(S/I^d, S) = \mathrm{Ext}^1_S(S/I^d, S) = 0$ and $\mathrm{Ext}^{j+1}_S(S/I^d, S) = \mathrm{Ext}^j_S(I^d, S)$.

We then proceed to compute the modules $\mathrm{Ext}^\bullet_S(I^d, S)$ (see Exercise 8.6). We write $V = \mathbb{C}^m$ and $W = \mathbb{C}^n$, and consider $X = \mathbb{G}(n, V)$ the Grassmannian of n dimensional quotients of V, with the tautological sequence

$$0 \longrightarrow \mathcal{R} \longrightarrow V \otimes \mathcal{O}_X \longrightarrow \mathcal{Q} \longrightarrow 0$$

where $\mathrm{rank}(\mathcal{R}) = m - n$ and $\mathrm{rank}(\mathcal{Q}) = n$. Tensoring the above exact sequence with W, we obtain

$$0 \longrightarrow \mathcal{R} \otimes W \longrightarrow V \otimes W \otimes \mathcal{O}_X \longrightarrow \mathcal{Q} \otimes W \longrightarrow 0,$$

which is the sequence (8.1) to which we apply the methods of Section 8. We have $\xi = \mathcal{R} \otimes W$, $U = V \otimes W$, $\eta = \mathcal{Q} \otimes W$. For every $d \geq 0$ we consider the locally free sheaf $\mathcal{V}_d$ on X defined by

$$\mathcal{V}_d = \det(\mathcal{Q})^{\otimes d} \otimes \det(W)^{\otimes d}.$$

Exercise 9.3. Show that $\mathcal{M}(\mathcal{V}_d)$ satisfies the hypothesis of Theorem 8.4 and that $M(\mathcal{V}_d) = I^d$.

We can now apply Theorem 8.4. Using the fact that $k = \mathrm{rank}(\xi) = n(m-n) = d_X$ is the dimension of X, we get

$$\mathrm{Ext}^j_S(I^d, S) = H^{d_X - j}(X, \mathcal{M}^\vee(\mathcal{V}_d))^* \otimes \det(V^* \otimes W^*),$$

and using Serre duality and some easy manipulations

$$\mathrm{Ext}^j_S(I^d, S) = H^j(X, \det(\mathcal{Q})^{\otimes(n-m-d)} \otimes \det(W)^{\otimes n-m-d} \otimes \mathrm{Sym}(\mathcal{Q} \otimes W)).$$

Exercise 9.4. Use Bott's theorem to show that $\mathrm{Ext}^j_S(I^d, S) = 0$ unless $j = s \cdot (m-n)$ for some $s = 0, \cdots, n$. Moreover, for each $s = 0, \cdots, n$ show that

$$\mathrm{Ext}^{s(m-n)}_S(I^d, S) = \bigoplus_{\substack{\lambda_{n-s} \geq -s,\ \lambda_{n-s+1} \leq n-m-s \\ \lambda_n \geq n-m-d}} S_{\lambda(n-s)} V \otimes S_\lambda W.$$

Exercise 9.5. Assuming that the maps in the directed system from (9.1) are injective (see the proof of [RWW14, Theorem 4.5] for an explanation of the injectivity), finish the calculation of $H^\bullet_I(S)$.

9.2. **Proof of Theorem 2.7 in the case when $s = n-1$.** We write as before $V = \mathbb{C}^m$ and $W = \mathbb{C}^n$ and consider $X = \mathbb{P}W = \mathbb{G}(n-1, W)$, the projective space of lines in W, with the tautological sequence

$$0 \longrightarrow \mathcal{R} \longrightarrow W \otimes \mathcal{O}_X \longrightarrow \mathcal{Q} \longrightarrow 0,$$

where $\mathcal{R} = \mathcal{O}_X(-1)$ is a line bundle, and $\mathrm{rank}(\mathcal{Q}) = n-1$. We tensor with V and obtain

$$0 \longrightarrow V \otimes \mathcal{R} \longrightarrow V \otimes W \otimes \mathcal{O}_X \longrightarrow V \otimes \mathcal{Q} \longrightarrow 0,$$

which we think of as the sequence (8.1). We have $\xi = V \otimes \mathcal{R}$, $U = V \otimes W$, $\eta = V \otimes \mathcal{Q}$. The map $\pi : Y \to U^*$ in (8.2) yields a small resolution of the set of matrices of rank at most $(n-1)$, and this implies that the direct image $\int_\pi \mathcal{O}_Y$ of the $\mathcal{D}_Y$-module $\mathcal{O}_Y$ along π coincides with D_{n-1}. When we think of $\mathcal{O}_Y$ as a quasi-coherent sheaf on X, we make the identification $\mathcal{O}_Y = \mathcal{S} = \mathrm{Sym}(\eta)$. It follows from [Rai14, Corollary 2.10] that we have the following equalities in the Grothendieck group $\Gamma(\mathrm{GL})$ of admissible GL-representations:

$$[D_{n-1}]_{\Gamma(\mathrm{GL})} = \left[\int_\pi \mathcal{O}_Y\right]_{\Gamma(\mathrm{GL})} = \sum_{i=0}^{d_X} (-1)^{d_X - i} \cdot \chi(X, \Omega^i_X \otimes \mathrm{Sym}(\eta) \otimes \det(\xi^*) \otimes \mathrm{Sym}(\xi^*)). \tag{9.2}$$

To explain the notation in (9.2), we define the Euler characteristic $\chi(X, \mathcal{E}) \in \Gamma(\mathrm{GL})$ for a quasi-coherent GL-equivariant sheaf $\mathcal{E}$ on X, whose cohomology groups $H^\bullet(X, \mathcal{E})$ are admissible GL-representations, by

$$\chi(X, \mathcal{E}) = \sum_{j=0}^{d_X} (-1)^j [H^j(X, \mathcal{E})]_{\Gamma(\mathrm{GL})}.$$

Exercise 9.6. Use the methods of Section 8 to show that we have a formal equality

$$\sum_{i=0}^{d_X} (-1)^i \bigwedge^i (\mathcal{Q} \otimes \mathcal{R}^*) \otimes \mathrm{Sym}(V \otimes \mathcal{Q} \oplus V^* \otimes \mathcal{R}^*) = \sum_{\substack{\lambda = (\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq 0) \\ r \geq 0}} S_{\tilde{\lambda}(r)} V \otimes S_\lambda \mathcal{Q} \otimes \mathcal{R}^{-r}, \tag{9.3}$$

where $\tilde{\lambda}(r) = (\lambda_1, \cdots, \lambda_{n-1}, 0^{m-n}, -r)$. You should start by defining an appropriate Grothendieck group where the equality (9.3) makes sense!

Hint: You should think of $(V \otimes \mathcal{Q} \oplus V^* \otimes \mathcal{R}^*)$ as linear forms on a vector space of pairs $(A, \vec{v})$ with $A \in \mathrm{Hom}(V, \mathcal{Q}^*)$ an $(n-1) \times m$ matrix, and $\vec{v} \in \mathrm{Hom}(\mathcal{R}^*, V)$ a column vector with m entries. Then $\mathcal{Q} \otimes \mathcal{R}^*$ are the quadratic functions that pick out the entries of the product $A\vec{v}$, and the left-hand side of (9.3) is just the Euler characteristic of the Koszul complex on $\mathcal{Q}\otimes\mathcal{R}^*$. This Koszul complex is exact, and the right hand side of (9.3) corresponds to the coordinate ring of the variety of pairs $(A,\vec{v})$ with $A\vec{v}=0$.

Combining (9.2) and (9.3) yields after some manipulations

$$[D_{n-1}]_{\Gamma(\mathrm{GL})} = \sum_{\substack{\lambda=(\lambda_1\geq\cdots\geq\lambda_{n-1}\geq 0)\\ r\geq 0}} \chi(X, S_{\tilde{\lambda}(r)}V\otimes S_\lambda\mathcal{Q}\otimes\mathcal{R}^{-r+n-m})\otimes\det(V^*)\otimes\det(W^*), \tag{9.4}$$

which using Bott's theorem gives the formula (2.3) in the case when $s=n-1$.

Exercise 9.7. Verify that Bott's theorem applied to (9.4) yields the formula (2.3) in the case when $s=n-1$.

References

[ABW81] Kaan Akin, David A. Buchsbaum, and Jerzy Weyman, *Resolutions of determinantal ideals: the submaximal minors*, Adv. in Math. **39** (1981), no. 1, 1–30, DOI 10.1016/0001-8708(81)90055-4. MR605350 (82h:13011)

[Bri93] Michel Brion, *Sur les modules de covariants*, Ann. Sci. École Norm. Sup. (4) **26** (1993), no. 1, 1–21 (French, with English and French summaries). MR1209911 (95c:14062)

[BS90] Winfried Bruns and Roland Schwänzl, *The number of equations defining a determinantal variety*, Bull. London Math. Soc. **22** (1990), no. 5, 439–445, DOI 10.1112/blms/22.5.439. MR1082012 (91k:14035)

[BV88] Winfried Bruns and Udo Vetter, *Determinantal rings*, Lecture Notes in Mathematics, vol. 1327, Springer-Verlag, Berlin, 1988. MR953963 (89i:13001)

[Bud13] Nero Budur, *Bernstein-Sato polynomials and generalizations*, available at `https://www3.nd.edu/~nbudur/Rolduc/RolducNotes.pdf` (2013). Lecture notes: Summer School on Algebra, Algorithms, and Algebraic Analysis, Rolduc Abbey, Netherlands.

[BMS06] Nero Budur, Mircea Mustaţă, and Morihiko Saito, *Bernstein-Sato polynomials of arbitrary varieties*, Compos. Math. **142** (2006), no. 3, 779–797, DOI 10.1112/S0010437X06002193. MR2231202 (2007c:32036)

[CSS13] Sergio Caracciolo, Alan D. Sokal, and Andrea Sportiello, *Algebraic/combinatorial proofs of Cayley-type identities for derivatives of determinants and Pfaffians*, Adv. in Appl. Math. **50** (2013), no. 4, 474–594, DOI 10.1016/j.aam.2012.12.001. MR3032306

[dCMM14] Mark de Cataldo, Luca Migliorini, and Mircea Mustaţă, *Combinatorics and topology of proper toric maps*, arXiv **1407.3497** (2014).

[dCEP80] C. de Concini, David Eisenbud, and C. Procesi, *Young diagrams and determinantal varieties*, Invent. Math. **56** (1980), no. 2, 129–165, DOI 10.1007/BF01392548. MR558865 (81m:14034)

[GS] Daniel R. Grayson and Michael E. Stillman, *Macaulay 2, a software system for research in algebraic geometry*, Available at `http://www.math.uiuc.edu/Macaulay2/`.

[HE71] M. Hochster and John A. Eagon, *Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci*, Amer. J. Math. **93** (1971), 1020–1058. MR0302643 (46 #1787)

[HR74] Melvin Hochster and Joel L. Roberts, *Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay*, Advances in Math. **13** (1974), 115–175. MR0347810 (50 #311)

[HTT08] Ryoshi Hotta, Kiyoshi Takeuchi, and Toshiyuki Tanisaki, *D-modules, perverse sheaves, and representation theory*, Progress in Mathematics, vol. 236, Birkhäuser Boston, Inc., Boston, MA, 2008. Translated from the 1995 Japanese edition by Takeuchi. MR2357361 (2008k:32022)

[Las78] Alain Lascoux, *Syzygies des variétés déterminantales*, Adv. in Math. **30** (1978), no. 3, 202–237, DOI 10.1016/0001-8708(78)90037-3 (French). MR520233 (80j:14043)

[LSW13] Gennady Lyubeznik, Anurag Singh, and Uli Walther, *Local cohomology modules supported at determinantal ideals*, arXiv **1308.4182** (2013).

[Kim03] Tatsuo Kimura, *Introduction to prehomogeneous vector spaces*, Translations of Mathematical Monographs, vol. 215, American Mathematical Society, Providence, RI, 2003. Translated from the 1998 Japanese original by Makoto Nagura and Tsuyoshi Niitani and revised by the author. MR1944442 (2003k:11180)

[PS73] C. Peskine and L. Szpiro, *Dimension projective finie et cohomologie locale. Applications à la démonstration de conjectures de M. Auslander, H. Bass et A. Grothendieck*, Inst. Hautes Études Sci. Publ. Math. **42** (1973), 47–119 (French). MR0374130 (51 #10330)

[Rai14] Claudiu Raicu, *Characters of equivariant $\mathcal{D}$-modules on Veronese cones*, arXiv **1412.8148** (2014).

[Rai15] ______, *Characters of equivariant $\mathcal{D}$-modules on spaces of matrices*, in preparation (2015).

[RW14a] Claudiu Raicu and Jerzy Weyman, *Local cohomology with support in generic determinantal ideals*, Algebra Number Theory **8** (2014), no. 5, 1231–1257, DOI 10.2140/ant.2014.8.1231. MR3263142

[RW14b] ———, *The syzygies of some thickenings of determinantal varieties*, arXiv **1411.0151** (2014).

[RWW14] Claudiu Raicu, Jerzy Weyman, and Emily E. Witt, *Local cohomology with support in ideals of maximal minors and sub-maximal Pfaffians*, Adv. Math. **250** (2014), 596–610, DOI 10.1016/j.aim.2013.10.005. MR3122178

[Sta82] Richard P. Stanley, *Linear Diophantine equations and local cohomology*, Invent. Math. **68** (1982), no. 2, 175–193, DOI 10.1007/BF01394054. MR666158 (83m:10017)

[VdB91] Michel Van den Bergh, *Cohen-Macaulayness of modules of covariants*, Invent. Math. **106** (1991), no. 2, 389–409, DOI 10.1007/BF01243917. MR1128219 (92m:14063)

[VdB94] ———, *A converse to Stanley's conjecture for* Sl_2, Proc. Amer. Math. Soc. **121** (1994), no. 1, 47–51, DOI 10.2307/2160363. MR1181176 (94g:20062)

[VdB95] ———, *Modules of covariants*, 2 (Zürich, 1994), Birkhäuser, Basel, 1995, pp. 352–362. MR1403936 (98e:13012)

[VdB99] ———, *Local cohomology of modules of covariants*, Adv. Math. **144** (1999), no. 2, 161–220, DOI 10.1006/aima.1998.1809. MR1695237 (2000d:14051)

[Wey03] Jerzy Weyman, *Cohomology of vector bundles and syzygies*, Cambridge Tracts in Mathematics, vol. 149, Cambridge University Press, Cambridge, 2003. MR1988690 (2004d:13020)

[Wit12] Emily E. Witt, *Local cohomology with support in ideals of maximal minors*, Adv. Math. **231** (2012), no. 3-4, 1998–2012, DOI 10.1016/j.aim.2012.07.001. MR2964631