Understanding of the defining equations and syzygies via inner projections and generic initial ideals

Sijong Kwak (KAIST, Korea)

Syzygies of Algebraic Varieties Workshop for Graduate Students, Postdocs and Young Researchers,
Nov 20-22, 2015, the University of Illinois at Chicago(UIC)

- $X \subset \mathbb{P}(V)$, $V \subset H^0(\mathcal{O}_X(1))$: a nondegenerate, irreducible and reduced variety of $\dim(X) = n$ and $\mathrm{codim}(X) = e$ defined over $K = \overline{K}$ of char $(K) \geq 0$. If $V = H^0(\mathcal{O}_X(1))$, one says that X is a linearly normal embedding.
- R/I_X: the projective coordinate ring of X where $R = K[x_0, x_1, \ldots, x_{n+e}]$ is a coordinate ring of $\mathbb{P}(V)$ and $I_X = \bigoplus_{m \geq 0} H^0(\mathcal{I}_X(m))$ is the saturated ideal.
- There is a basic exact sequence:

$$0 \to R/I_X \to \bigoplus_{m \geq 0} H^0(\mathcal{O}_X(m)) \to \bigoplus_{m \geq 0} H^1(\mathcal{I}_X(m)) \to 0$$

and X is called "projectively normal" if $\bigoplus_{m \geq 0} H^1(\mathcal{I}_X(m)) = 0$.

- $X \subset \mathbb{P}(V)$, $V \subset H^0(\mathcal{O}_X(1))$: a nondegenerate, irreducible and reduced variety of $\dim(X) = n$ and $\mathrm{codim}(X) = e$ defined over $K = \overline{K}$ of char $(K) \geq 0$. If $V = H^0(\mathcal{O}_X(1))$, one says that X is a linearly normal embedding.

- R/I_X: the projective coordinate ring of X where $R = K[x_0, x_1, \ldots, x_{n+e}]$ is a coordinate ring of $\mathbb{P}(V)$ and $I_X = \bigoplus_{m \geq 0} H^0(\mathcal{I}_X(m))$ is the saturated ideal.

- There is a basic exact sequence:

$$0 \to R/I_X \to \bigoplus_{m \geq 0} H^0(\mathcal{O}_X(m)) \to \bigoplus_{m \geq 0} H^1(\mathcal{I}_X(m)) \to 0$$

and X is called "projectively normal" if $\bigoplus_{m \geq 0} H^1(\mathcal{I}_X(m)) = 0$.

- $X \subset \mathbb{P}(V)$, $V \subset H^0(\mathcal{O}_X(1))$: a nondegenerate, irreducible and reduced variety of $\dim(X) = n$ and $\mathrm{codim}(X) = e$ defined over $K = \overline{K}$ of char $(K) \geq 0$. If $V = H^0(\mathcal{O}_X(1))$, one says that X is a linearly normal embedding.
- R/I_X: the projective coordinate ring of X where $R = K[x_0, x_1, \ldots, x_{n+e}]$ is a coordinate ring of $\mathbb{P}(V)$ and $I_X = \bigoplus_{m \geq 0} H^0(\mathcal{I}_X(m))$ is the saturated ideal.
- There is a basic exact sequence:

$$0 \to R/I_X \to \bigoplus_{m \geq 0} H^0(\mathcal{O}_X(m)) \to \bigoplus_{m \geq 0} H^1(\mathcal{I}_X(m)) \to 0$$

and X is called "projectively normal" if $\bigoplus_{m \geq 0} H^1(\mathcal{I}_X(m)) = 0$.

- $X \subset \mathbb{P}(V)$, $V \subset H^0(\mathcal{O}_X(1))$: a nondegenerate, irreducible and reduced variety of $\dim(X) = n$ and $\mathrm{codim}(X) = e$ defined over $K = \overline{K}$ of char $(K) \geq 0$. If $V = H^0(\mathcal{O}_X(1))$, one says that X is a linearly normal embedding.
- R/I_X: the projective coordinate ring of X where $R = K[x_0, x_1, \ldots, x_{n+e}]$ is a coordinate ring of $\mathbb{P}(V)$ and $I_X = \bigoplus_{m \geq 0} H^0(\mathcal{I}_X(m))$ is the saturated ideal.
- There is a basic exact sequence:

$$0 \to R/I_X \to \bigoplus_{m \geq 0} H^0(\mathcal{O}_X(m)) \to \bigoplus_{m \geq 0} H^1(\mathcal{I}_X(m)) \to 0$$

and X is called "projectively normal" if $\bigoplus_{m \geq 0} H^1(\mathcal{I}_X(m)) = 0$.

- $X \subset \mathbb{P}(V)$, $V \subset H^0(\mathcal{O}_X(1))$: a nondegenerate, irreducible and reduced variety of $\dim(X) = n$ and $\mathrm{codim}(X) = e$ defined over $K = \overline{K}$ of char $(K) \geq 0$. If $V = H^0(\mathcal{O}_X(1))$, one says that X is a linearly normal embedding.
- R/I_X: the projective coordinate ring of X where $R = K[x_0, x_1, \ldots, x_{n+e}]$ is a coordinate ring of $\mathbb{P}(V)$ and $I_X = \bigoplus_{m \geq 0} H^0(\mathcal{I}_X(m))$ is the saturated ideal.
- There is a basic exact sequence:

$$0 \to R/I_X \to \bigoplus_{m\geq 0} H^0(\mathcal{O}_X(m)) \to \bigoplus_{m\geq 0} H^1(\mathcal{I}_X(m)) \to 0$$

and X is called "projectively normal" if $\bigoplus_{m\geq 0} H^1(\mathcal{I}_X(m)) = 0$.

- $X \subset \mathbb{P}(V)$, $V \subset H^0(\mathcal{O}_X(1))$: a nondegenerate, irreducible and reduced variety of $\dim(X) = n$ and $\mathrm{codim}(X) = e$ defined over $K = \overline{K}$ of char $(K) \geq 0$. If $V = H^0(\mathcal{O}_X(1))$, one says that X is a linearly normal embedding.
- R/I_X: the projective coordinate ring of X where $R = K[x_0, x_1, \ldots, x_{n+e}]$ is a coordinate ring of $\mathbb{P}(V)$ and $I_X = \bigoplus_{m \geq 0} H^0(\mathcal{I}_X(m))$ is the saturated ideal.
- There is a basic exact sequence:

$$0 \to R/I_X \to \bigoplus_{m \geq 0} H^0(\mathcal{O}_X(m)) \to \bigoplus_{m \geq 0} H^1(\mathcal{I}_X(m)) \to 0$$

and X is called "projectively normal" if $\bigoplus_{m \geq 0} H^1(\mathcal{I}_X(m)) = 0$.

- $X \subset \mathbb{P}(V)$, $V \subset H^0(\mathcal{O}_X(1))$: a nondegenerate, irreducible and reduced variety of $\dim(X) = n$ and $\mathrm{codim}(X) = e$ defined over $K = \overline{K}$ of char $(K) \geq 0$. If $V = H^0(\mathcal{O}_X(1))$, one says that X is a linearly normal embedding.

- R/I_X: the projective coordinate ring of X where $R = K[x_0, x_1, \ldots, x_{n+e}]$ is a coordinate ring of $\mathbb{P}(V)$ and $I_X = \bigoplus_{m \geq 0} H^0(\mathcal{I}_X(m))$ is the saturated ideal.

- There is a basic exact sequence:

$$0 \to R/I_X \to \bigoplus_{m \geq 0} H^0(\mathcal{O}_X(m)) \to \bigoplus_{m \geq 0} H^1(\mathcal{I}_X(m)) \to 0$$

and X is called "projectively normal" if $\bigoplus_{m \geq 0} H^1(\mathcal{I}_X(m)) = 0$.

- **Basic Goal**: Understand the minimal free resolutions of R/I_X and $R(X) = \bigoplus_{m \geq 0} H^0(\mathcal{O}_X(m))$ and their associated Betti tables in terms of geometric invariants.
- Defining equations and their relations (called "syzygies") of X appear in the (unique) minimal free resolution of R/I_X: $\cdots \to L_i \to L_{i-1} \to \cdots \to L_1 \to R \to R/I_X \to 0$ where $L_i = \bigoplus_j R(-i-j)^{\beta_{i,j}(X)}$.
- Note that $\beta_{i,j}(X)$ is the rank of the degree $i+j$ part in L_i and $\beta_{i,j}(X) = \dim_K \operatorname{Tor}_i^R(R/I_X, K)_{i+j}$.
- The simplest nontrivial example is a rational normal curve $\upsilon_d(\mathbb{P}^1) \hookrightarrow \mathbb{P}^d$. How to compute the minimal free resolution of $\upsilon_d(\mathbb{P}^1) \subset \mathbb{P}^d$ for $d = 3, 4$ by hand or by using Macaulay 2 ?

- **Basic Goal**: Understand the minimal free resolutions of R/I_X and $R(X) = \bigoplus_{m\geq 0} H^0(\mathcal{O}_X(m))$ and their associated Betti tables in terms of geometric invariants.

- Defining equations and their relations (called "syzygies") of X appear in the (unique) minimal free resolution of R/I_X:
$\cdots \to L_i \to L_{i-1} \to \cdots \to L_1 \to R \to R/I_X \to 0$ where
$L_i = \bigoplus_j R(-i-j)^{\beta_{i,j}(X)}$.

- Note that $\beta_{i,j}(X)$ is the rank of the degree $i+j$ part in L_i and $\beta_{i,j}(X) = \dim_K \mathrm{Tor}_i^R(R/I_X, K)_{i+j}$.

- The simplest nontrivial example is a rational normal curve $\upsilon_d(\mathbb{P}^1) \hookrightarrow \mathbb{P}^d$. How to compute the minimal free resolution of $\upsilon_d(\mathbb{P}^1) \subset \mathbb{P}^d$ for $d = 3, 4$ by hand or by using Macaulay 2 ?

- **Basic Goal**: Understand the minimal free resolutions of R/I_X and $R(X) = \bigoplus_{m \geq 0} H^0(\mathcal{O}_X(m))$ and their associated Betti tables in terms of geometric invariants.
- Defining equations and their relations (called "syzygies") of X appear in the (unique) minimal free resolution of R/I_X: $\cdots \to L_i \to L_{i-1} \to \cdots \to L_1 \to R \to R/I_X \to 0$ where $L_i = \bigoplus_j R(-i-j)^{\beta_{i,j}(X)}$.
- Note that $\beta_{i,j}(X)$ is the rank of the degree $i+j$ part in L_i and $\beta_{i,j}(X) = \dim_K \operatorname{Tor}_i^R(R/I_X, K)_{i+j}$.
- The simplest nontrivial example is a rational normal curve $\upsilon_d(\mathbb{P}^1) \hookrightarrow \mathbb{P}^d$. How to compute the minimal free resolution of $\upsilon_d(\mathbb{P}^1) \subset \mathbb{P}^d$ for $d = 3, 4$ by hand or by using Macaulay 2 ?

- **Basic Goal**: Understand the minimal free resolutions of R/I_X and $R(X) = \bigoplus_{m\geq 0} H^0(\mathcal{O}_X(m))$ and their associated Betti tables in terms of geometric invariants.

- Defining equations and their relations (called "syzygies") of X appear in the (unique) minimal free resolution of R/I_X:
$\cdots \to L_i \to L_{i-1} \to \cdots \to L_1 \to R \to R/I_X \to 0$ where
$L_i = \bigoplus_j R(-i-j)^{\beta_{i,j}(X)}$.

- Note that $\beta_{i,j}(X)$ is the rank of the degree $i+j$ part in L_i and $\beta_{i,j}(X) = \dim_K \mathrm{Tor}_i^R(R/I_X, K)_{i+j}$.

- The simplest nontrivial example is a rational normal curve $\upsilon_d(\mathbb{P}^1) \hookrightarrow \mathbb{P}^d$. How to compute the minimal free resolution of $\upsilon_d(\mathbb{P}^1) \subset \mathbb{P}^d$ for $d = 3, 4$ by hand or by using Macaulay 2 ?

- **Basic Goal**: Understand the minimal free resolutions of R/I_X and $R(X) = \bigoplus_{m\geq 0} H^0(\mathcal{O}_X(m))$ and their associated Betti tables in terms of geometric invariants.

- Defining equations and their relations (called "syzygies") of X appear in the (unique) minimal free resolution of R/I_X:
$\cdots \to L_i \to L_{i-1} \to \cdots \to L_1 \to R \to R/I_X \to 0$ where
$L_i = \bigoplus_j R(-i-j)^{\beta_{i,j}(X)}$.

- Note that $\beta_{i,j}(X)$ is the rank of the degree $i+j$ part in L_i and $\beta_{i,j}(X) = \dim_K \mathrm{Tor}_i^R(R/I_X, K)_{i+j}$.
- The simplest nontrivial example is a rational normal curve $\upsilon_d(\mathbb{P}^1) \hookrightarrow \mathbb{P}^d$. How to compute the minimal free resolution of $\upsilon_d(\mathbb{P}^1) \subset \mathbb{P}^d$ for $d = 3, 4$ by hand or by using Macaulay 2 ?

- The section ring $R(X) := \bigoplus_{m\geq 0} H^0(\mathcal{O}_X(m))$ has also the following (unique) minimal free resolution as a graded R-module:

$$\cdots \to L_i \to \cdots \to L_1 \to L_0 \to \bigoplus_{m\geq 0} H^0(\mathcal{O}_X(m)) \to 0$$

where $L_0 = R \oplus R(-1)^t \bigoplus_{j\geq 2} R(-j)^{\beta_{0,j}}$.

- Note that $t = \mathrm{codim}(V, H^0(\mathcal{O}_X(1))$ and if X is not linearly normal, the basis elements in $H^0(\mathcal{O}_X(1)) \setminus V$ should be generators of $R(X)$ in degree 1.

- In particular, the minimal free resolution of the section ring $R(X)$ also encodes some geometric information on the embedding $X \hookrightarrow \mathbb{P}(V)$.

- The section ring $R(X) := \bigoplus_{m \geq 0} H^0(\mathcal{O}_X(m))$ has also the following (unique) minimal free resolution as a graded R-module:

$$\cdots \to L_i \to \cdots \to L_1 \to L_0 \to \bigoplus_{m \geq 0} H^0(\mathcal{O}_X(m)) \to 0$$

  where $L_0 = R \oplus R(-1)^t \bigoplus_{j \geq 2} R(-j)^{\beta_{0,j}}$.

- Note that $t = \mathrm{codim}(V, H^0(\mathcal{O}_X(1))$ and if X is not linearly normal, the basis elements in $H^0(\mathcal{O}_X(1)) \setminus V$ should be generators of $R(X)$ in degree 1.

- In particular, the minimal free resolution of the section ring $R(X)$ also encodes some geometric information on the embedding $X \hookrightarrow \mathbb{P}(V)$.

- The section ring $R(X) := \bigoplus_{m\geq 0} H^0(\mathcal{O}_X(m))$ has also the following (unique) minimal free resolution as a graded R-module:

$$\cdots \to L_i \to \cdots \to L_1 \to L_0 \to \bigoplus_{m\geq 0} H^0(\mathcal{O}_X(m)) \to 0$$

where $L_0 = R \oplus R(-1)^t \bigoplus_{j\geq 2} R(-j)^{\beta_{0,j}}$.

- Note that $t = \operatorname{codim}(V, H^0(\mathcal{O}_X(1))$ and if X is not linearly normal, the basis elements in $H^0(\mathcal{O}_X(1)) \setminus V$ should be generators of $R(X)$ in degree 1.

- In particular, the minimal free resolution of the section ring $R(X)$ also encodes some geometric information on the embedding $X \hookrightarrow \mathbb{P}(V)$.

- The section ring $R(X) := \bigoplus_{m \geq 0} H^0(\mathcal{O}_X(m))$ has also the following (unique) minimal free resolution as a graded R-module:

$$\cdots \to L_i \to \cdots \to L_1 \to L_0 \to \bigoplus_{m \geq 0} H^0(\mathcal{O}_X(m)) \to 0$$

  where $L_0 = R \oplus R(-1)^t \bigoplus_{j \geq 2} R(-j)^{\beta_{0,j}}$.

- Note that $t = \operatorname{codim}(V, H^0(\mathcal{O}_X(1))$ and if X is not linearly normal, the basis elements in $H^0(\mathcal{O}_X(1)) \setminus V$ should be generators of $R(X)$ in degree 1.

- In particular, the minimal free resolution of the section ring $R(X)$ also encodes some geometric information on the embedding $X \hookrightarrow \mathbb{P}(V)$.

Geometric information of the section ring

For example, if the section ring $\bigoplus_{m\geq 0} H^0(\mathcal{O}_X(m))$ of a smooth variety X has the following minimal free resolution:

$$\cdots \to R(-2)^{\beta_{1,1}} \to R \oplus R(-1)^t \to \bigoplus_{m\geq 0} H^0(\mathcal{O}_X(m)) \to 0,$$

i.e. as simple as possible up to the first syzygies,
then we have the following properties (due to E. Park-K, 2005):

- X is k-normal for all $k \geq t+1$;
- X is cut out by equations of degree at most $t+2$;
- $\mathrm{reg}(X) \leq \max\{m+1, t+2\}$ where $m = \mathrm{reg}(\mathcal{O}_X)$.

Note that if $t = 0$, then X is projectively normal, and so it is trivial.

Geometric information of the section ring

For example, if the section ring $\bigoplus_{m\geq 0} H^0(\mathcal{O}_X(m))$ of a smooth variety X has the following minimal free resolution:

$$\cdots \to R(-2)^{\beta_{1,1}} \to R \oplus R(-1)^t \to \bigoplus_{m\geq 0} H^0(\mathcal{O}_X(m)) \to 0,$$

i.e. as simple as possible up to the first syzygies,
then we have the following properties (due to E. Park-K, 2005):

- X is k-normal for all $k \geq t+1$;
- X is cut out by equations of degree at most $t+2$;
- $\mathrm{reg}(X) \leq \max\{m+1, t+2\}$ where $m = \mathrm{reg}(\mathcal{O}_X)$.

Note that if $t = 0$, then X is projectively normal, and so it is trivial.

Geometric information of the section ring

For example, if the section ring $\bigoplus_{m\geq 0} H^0(\mathcal{O}_X(m))$ of a smooth variety X has the following minimal free resolution:

$$\cdots \to R(-2)^{\beta_{1,1}} \to R \oplus R(-1)^t \to \bigoplus_{m\geq 0} H^0(\mathcal{O}_X(m)) \to 0,$$

i.e. as simple as possible up to the first syzygies,
then we have the following properties (due to E. Park-K, 2005):

- X is k-normal for all $k \geq t+1$;
- X is cut out by equations of degree at most $t+2$;
- $\mathrm{reg}(X) \leq \max\{m+1, t+2\}$ where $m = \mathrm{reg}(\mathcal{O}_X)$.

Note that if $t = 0$, then X is projectively normal, and so it is trivial.

Geometric information of the section ring

For example, if the section ring $\bigoplus_{m\geq 0} H^0(\mathcal{O}_X(m))$ of a smooth variety X has the following minimal free resolution:

$$\cdots \to R(-2)^{\beta_{1,1}} \to R \oplus R(-1)^t \to \bigoplus_{m\geq 0} H^0(\mathcal{O}_X(m)) \to 0,$$

i.e. as simple as possible up to the first syzygies,
then we have the following properties (due to E. Park-K, 2005):

- X is k-normal for all $k \geq t+1$;
- X is cut out by equations of degree at most $t+2$;
- $\mathrm{reg}(X) \leq \max\{m+1, t+2\}$ where $m = \mathrm{reg}(\mathcal{O}_X)$.

Note that if $t = 0$, then X is projectively normal, and so it is trivial.

Geometric information of the section ring

For example, if the section ring $\bigoplus_{m\geq 0} H^0(\mathcal{O}_X(m))$ of a smooth variety X has the following minimal free resolution:

$$\cdots \to R(-2)^{\beta_{1,1}} \to R \oplus R(-1)^t \to \bigoplus_{m\geq 0} H^0(\mathcal{O}_X(m)) \to 0,$$

i.e. as simple as possible up to the first syzygies,
then we have the following properties (due to E. Park-K, 2005):

- X is k-normal for all $k \geq t+1$;
- X is cut out by equations of degree at most $t+2$;
- $\mathrm{reg}(X) \leq \max\{m+1, t+2\}$ where $m = \mathrm{reg}(\mathcal{O}_X)$.

Note that if $t = 0$, then X is projectively normal, and so it is trivial.

Geometric information of the section ring

For example, if the section ring $\bigoplus_{m\geq 0} H^0(\mathcal{O}_X(m))$ of a smooth variety X has the following minimal free resolution:

$$\cdots \to R(-2)^{\beta_{1,1}} \to R \oplus R(-1)^t \to \bigoplus_{m\geq 0} H^0(\mathcal{O}_X(m)) \to 0,$$

i.e. as simple as possible up to the first syzygies,
then we have the following properties (due to E. Park-K, 2005):

- X is k-normal for all $k \geq t+1$;
- X is cut out by equations of degree at most $t+2$;
- $\mathrm{reg}(X) \leq \max\{m+1, t+2\}$ where $m = \mathrm{reg}(\mathcal{O}_X)$.

Note that if $t = 0$, then X is projectively normal, and so it is trivial.

Geometric information of the section ring

For example, if the section ring $\bigoplus_{m\geq 0} H^0(\mathcal{O}_X(m))$ of a smooth variety X has the following minimal free resolution:

$$\cdots \to R(-2)^{\beta_{1,1}} \to R \oplus R(-1)^t \to \bigoplus_{m\geq 0} H^0(\mathcal{O}_X(m)) \to 0,$$

i.e. as simple as possible up to the first syzygies,
then we have the following properties (due to E. Park-K, 2005):

- X is k-normal for all $k \geq t+1$;
- X is cut out by equations of degree at most $t+2$;
- $\mathrm{reg}(X) \leq \max\{m+1, t+2\}$ where $m = \mathrm{reg}(\mathcal{O}_X)$.

Note that if $t = 0$, then X is projectively normal, and so it is trivial.

Let $X \subset \mathbb{P}^{n+e}$ be a variety of $\dim(X) = n$ and $\deg(X) = d$. We have the Betti table of R/I_X associated to the minimal free resolution.

	0	1	2	3	$\cdots$	$i-1$	i	$i+1$	$\cdots$	$\triangle$
0	1	$-$	$-$	$-$	$\cdots$	$-$	$-$	$-$	$\cdots$	$-$
1	$-$	$\beta_{1,1}$	$\beta_{2,1}$	$\beta_{3,1}$	$\cdots$	$\beta_{i-1,1}$	$\beta_{i,1}$	$\beta_{i+1,1}$	$\cdots$	$\beta_{\triangle,1}$
2	$-$	$\beta_{1,2}$	$\beta_{2,2}$	$\beta_{3,2}$	$\cdots$	$\beta_{i-1,2}$	$\beta_{i,2}$	$\beta_{i+1,2}$	$\cdots$	$\beta_{\triangle,2}$
$\vdots$	$-$	$-$	$\cdots$	$-$	$\ddots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\ddots$
j	$-$	$\beta_{1,j}$	$\beta_{2,j}$	$\beta_{3,j}$	$\cdots$	$\beta_{i-1,j}$	$\beta_{i,j}$	$\beta_{i+1,j}$	$\cdots$	$\beta_{\triangle,j}$
$\vdots$			$\cdots$	$-$	$\ddots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\ddots$
$\square$	$-$	$\beta_{1,\square}$	$\beta_{2,\square}$	$\beta_{3,\square}$	$\cdots$	$\beta_{i-1,\square}$	$\beta_{i,\square}$	$\beta_{i+1,\square}$	$\cdots$	$\beta_{\triangle,\square}$

$\triangle =$ the projective dimension of R/I_X, $\triangle \geq e$.

$\square = \mathrm{reg}(R/I_X) = \mathrm{reg}(X) - 1$, where

$\mathrm{reg}(X) := \min\{\alpha \mid H^i(\mathbb{P}, \mathcal{I}_X(\alpha - i)) = 0\}$.

(Eisenbud-Goto Conjecture) $\square \leq \deg(X) - \mathrm{codim}(X) = d - e$.

Let $X \subset \mathbb{P}^{n+e}$ be a variety of $\dim(X) = n$ and $\deg(X) = d$. We have the Betti table of R/I_X associated to the minimal free resolution.

	0	1	2	3	$\cdots$	$i-1$	i	$i+1$	$\cdots$	$\triangle$
0	1	–	–	–	$\cdots$	–	–	–	$\cdots$	–
1	–	$\beta_{1,1}$	$\beta_{2,1}$	$\beta_{3,1}$	$\cdots$	$\beta_{i-1,1}$	$\beta_{i,1}$	$\beta_{i+1,1}$	$\cdots$	$\beta_{\triangle,1}$
2	–	$\beta_{1,2}$	$\beta_{2,2}$	$\beta_{3,2}$	$\cdots$	$\beta_{i-1,2}$	$\beta_{i,2}$	$\beta_{i+1,2}$	$\cdots$	$\beta_{\triangle,2}$
$\vdots$	–	–	$\cdots$	–	$\ddots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\ddots$
j	–	$\beta_{1,j}$	$\beta_{2,j}$	$\beta_{3,j}$	$\cdots$	$\beta_{i-1,j}$	$\beta_{i,j}$	$\beta_{i+1,j}$	$\cdots$	$\beta_{\triangle,j}$
$\vdots$			$\cdots$	–	$\ddots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\ddots$
$\square$	–	$\beta_{1,\square}$	$\beta_{2,\square}$	$\beta_{3,\square}$	$\cdots$	$\beta_{i-1,\square}$	$\beta_{i,\square}$	$\beta_{i+1,\square}$	$\cdots$	$\beta_{\triangle,\square}$

$\triangle =$ the projective dimension of R/I_X, $\triangle \geq e$.

$\square = \mathrm{reg}(R/I_X) = \mathrm{reg}(X) - 1$, where

$\mathrm{reg}(X) := \min\{\alpha \mid H^i(\mathbb{P}, \mathcal{I}_X(\alpha - i)) = 0\}$.

(Eisenbud-Goto Conjecture) $\square \leq \deg(X) - \mathrm{codim}(X) = d - e$.

Let $X \subset \mathbb{P}^{n+e}$ be a variety of $\dim(X) = n$ and $\deg(X) = d$. We have the Betti table of R/I_X associated to the minimal free resolution.

	0	1	2	3	$\cdots$	$i-1$	i	$i+1$	$\cdots$	$\triangle$
0	1	–	–	–	$\cdots$	–	–	–	$\cdots$	–
1	–	$\beta_{1,1}$	$\beta_{2,1}$	$\beta_{3,1}$	$\cdots$	$\beta_{i-1,1}$	$\beta_{i,1}$	$\beta_{i+1,1}$	$\cdots$	$\beta_{\triangle,1}$
2	–	$\beta_{1,2}$	$\beta_{2,2}$	$\beta_{3,2}$	$\cdots$	$\beta_{i-1,2}$	$\beta_{i,2}$	$\beta_{i+1,2}$	$\cdots$	$\beta_{\triangle,2}$
$\vdots$	–	–	$\cdots$	–	$\ddots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\ddots$
j	–	$\beta_{1,j}$	$\beta_{2,j}$	$\beta_{3,j}$	$\cdots$	$\beta_{i-1,j}$	$\beta_{i,j}$	$\beta_{i+1,j}$	$\cdots$	$\beta_{\triangle,j}$
$\vdots$			$\cdots$	–	$\ddots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\ddots$
$\square$	–	$\beta_{1,\square}$	$\beta_{2,\square}$	$\beta_{3,\square}$	$\cdots$	$\beta_{i-1,\square}$	$\beta_{i,\square}$	$\beta_{i+1,\square}$	$\cdots$	$\beta_{\triangle,\square}$

$\triangle =$ the projective dimension of R/I_X, $\triangle \geq e$.

$\square = \mathrm{reg}(R/I_X) = \mathrm{reg}(X) - 1$, where

$\mathrm{reg}(X) := \min \{\alpha \mid H^i(\mathbb{P}, \mathcal{I}_X(\alpha - i)) = 0\}$.

(Eisenbud-Goto Conjecture) $\square \leq \deg(X) - \mathrm{codim}(X) = d - e$.

Let $X \subset \mathbb{P}^{n+e}$ be a variety of $\dim(X) = n$ and $\deg(X) = d$. We have the Betti table of R/I_X associated to the minimal free resolution.

	0	1	2	3	$\cdots$	$i-1$	i	$i+1$	$\cdots$	$\triangle$
0	1	–	–	–	$\cdots$	–	–	–	$\cdots$	–
1	–	$\beta_{1,1}$	$\beta_{2,1}$	$\beta_{3,1}$	$\cdots$	$\beta_{i-1,1}$	$\beta_{i,1}$	$\beta_{i+1,1}$	$\cdots$	$\beta_{\triangle,1}$
2	–	$\beta_{1,2}$	$\beta_{2,2}$	$\beta_{3,2}$	$\cdots$	$\beta_{i-1,2}$	$\beta_{i,2}$	$\beta_{i+1,2}$	$\cdots$	$\beta_{\triangle,2}$
$\vdots$	–	–	$\ldots$	–	$\ddots$	$\ldots$	$\vdots$	$\vdots$	$\ldots$	$\ddots$
j	–	$\beta_{1,j}$	$\beta_{2,j}$	$\beta_{3,j}$	$\cdots$	$\beta_{i-1,j}$	$\beta_{i,j}$	$\beta_{i+1,j}$	$\cdots$	$\beta_{\triangle,j}$
$\vdots$			$\ldots$	–	$\ddots$	$\ldots$	$\vdots$	$\vdots$	$\ldots$	$\ddots$
$\square$	–	$\beta_{1,\square}$	$\beta_{2,\square}$	$\beta_{3,\square}$	$\cdots$	$\beta_{i-1,\square}$	$\beta_{i,\square}$	$\beta_{i+1,\square}$	$\cdots$	$\beta_{\triangle,\square}$

$\triangle =$ the projective dimension of R/I_X, $\triangle \geq e$.

$\square = \mathrm{reg}(R/I_X) = \mathrm{reg}(X) - 1$, where

$$\mathrm{reg}(X) := \min\{\alpha \mid H^i(\mathbb{P}, \mathcal{I}_X(\alpha - i)) = 0\}.$$

(Eisenbud-Goto Conjecture) $\square \leq \deg(X) - \mathrm{codim}(X) = d - e$.

Let $X \subset \mathbb{P}^{n+e}$ be a variety of $\dim(X) = n$ and $\deg(X) = d$. We have the Betti table of R/I_X associated to the minimal free resolution.

	0	1	2	3	$\cdots$	$i-1$	i	$i+1$	$\cdots$	$\triangle$
0	1	–	–	–	$\cdots$	–	–	–	$\cdots$	–
1	–	$\beta_{1,1}$	$\beta_{2,1}$	$\beta_{3,1}$	$\cdots$	$\beta_{i-1,1}$	$\beta_{i,1}$	$\beta_{i+1,1}$	$\cdots$	$\beta_{\triangle,1}$
2	–	$\beta_{1,2}$	$\beta_{2,2}$	$\beta_{3,2}$	$\cdots$	$\beta_{i-1,2}$	$\beta_{i,2}$	$\beta_{i+1,2}$	$\cdots$	$\beta_{\triangle,2}$
$\vdots$	–	–	$\dots$	–	$\ddots$	$\dots$	$\vdots$	$\vdots$	$\dots$	$\ddots$
j	–	$\beta_{1,j}$	$\beta_{2,j}$	$\beta_{3,j}$	$\cdots$	$\beta_{i-1,j}$	$\beta_{i,j}$	$\beta_{i+1,j}$	$\cdots$	$\beta_{\triangle,j}$
$\vdots$			$\dots$	–	$\ddots$	$\dots$	$\vdots$	$\vdots$	$\dots$	$\ddots$
$\square$	–	$\beta_{1,\square}$	$\beta_{2,\square}$	$\beta_{3,\square}$	$\cdots$	$\beta_{i-1,\square}$	$\beta_{i,\square}$	$\beta_{i+1,\square}$	$\cdots$	$\beta_{\triangle,\square}$

$\triangle$ = the projective dimension of R/I_X, $\triangle \geq e$.

$\square = \mathrm{reg}(R/I_X) = \mathrm{reg}(X) - 1$, where

$\mathrm{reg}(X) := \min\{\alpha \mid H^i(\mathbb{P}, \mathcal{I}_X(\alpha - i)) = 0\}$.

(Eisenbud-Goto Conjecture) $\square \leq \deg(X) - \mathrm{codim}(X) = d - e$.

By the symmetry of Tor, the graded Betti numbers are also defined via the Koszul exact sequence of the base field K:

$V = K\langle x_0, \cdots, x_{n+e}\rangle$ be the K-vector space in $K[x_0, \ldots, x_{n+e}]$.
Then, $\mathrm{Tor}_i^R(R/I_X, K)_{i+j}$ is the homology of the Koszul complex:

$$\wedge^{i+1} V \otimes (R/I_X)_{j-1} \xrightarrow{\varphi_{i+1,j-1}} \wedge^i V \otimes (R/I_X)_j \xrightarrow{\varphi_{i,j}} \wedge^{i-1} V \otimes (R/I_X)_{j+1},$$

where the map is given by $\varphi_{i,j}(x_{\alpha_1} \wedge x_{\alpha_2} \wedge \cdots \wedge x_{\alpha_i} \otimes m) = \sum_{1\leq\mu\leq i} (-1)^{\mu-1} x_{\alpha_1} \cdots \wedge \hat{x_{\alpha_\mu}} \wedge \ldots \wedge x_{\alpha_i} \otimes (x_{\alpha_\mu} \cdot m)$.

Note that the Koszul complex is exact if $i > n+e+1$ or $j >> 0$.

- $\beta_{1,1}(X)$: the number of quadrics $Q_i \in I_X$;
- $\beta_{2,1}(X)$ is the number of linear relations of the form $\Sigma L_i Q_i = 0$;
- $\beta_{3,1}(X)$ is the number of linear relations of linear relations;
- $\beta_{1,2}(X)$ is the number of cubic generators of I_X.

By the symmetry of Tor, the graded Betti numbers are also defined via the Koszul exact sequence of the base field K:
$V = K\langle x_0, \cdots, x_{n+e}\rangle$ be the K-vector space in $K[x_0, \dots, x_{n+e}]$.
Then, $\mathrm{Tor}_i^R(R/I_X, K)_{i+j}$ is the homology of the Koszul complex:

$$\wedge^{i+1} V \otimes (R/I_X)_{j-1} \xrightarrow{\varphi_{i+1,j-1}} \wedge^{i} V \otimes (R/I_X)_{j} \xrightarrow{\varphi_{i,j}} \wedge^{i-1} V \otimes (R/I_X)_{j+1},$$

where the map is given by $\varphi_{i,j}(x_{\alpha_1} \wedge x_{\alpha_2} \wedge \cdots \wedge x_{\alpha_i} \otimes m) = \sum_{1 \le \mu \le i} (-1)^{\mu-1} x_{\alpha_1} \cdots \wedge \hat{x_{\alpha_\mu}} \wedge \ldots \wedge x_{\alpha_i} \otimes (x_{\alpha_\mu} \cdot m)$.

Note that the Koszul complex is exact if $i > n + e + 1$ or $j >> 0$.

- $\beta_{1,1}(X)$: the number of quadrics $Q_i \in I_X$;
- $\beta_{2,1}(X)$ is the number of linear relations of the form $\Sigma L_i Q_i = 0$;
- $\beta_{3,1}(X)$ is the number of linear relations of linear relations;
- $\beta_{1,2}(X)$ is the number of cubic generators of I_X.

By the symmetry of Tor, the graded Betti numbers are also defined via the Koszul exact sequence of the base field K:
$V = K\langle x_0, \cdots, x_{n+e}\rangle$ be the K-vector space in $K[x_0, \ldots, x_{n+e}]$.
Then, $\mathrm{Tor}_i^R(R/I_X, K)_{i+j}$ is the homology of the Koszul complex:

$$\wedge^{i+1} V \otimes (R/I_X)_{j-1} \xrightarrow{\varphi_{i+1,j-1}} \wedge^i V \otimes (R/I_X)_j \xrightarrow{\varphi_{i,j}} \wedge^{i-1} V \otimes (R/I_X)_{j+1},$$

where the map is given by $\varphi_{i,j}(x_{\alpha_1} \wedge x_{\alpha_2} \wedge \cdots \wedge x_{\alpha_i} \otimes m) = \sum_{1 \le \mu \le i} (-1)^{\mu-1} x_{\alpha_1} \cdots \wedge \hat{x_{\alpha_\mu}} \wedge \ldots \wedge x_{\alpha_i} \otimes (x_{\alpha_\mu} \cdot m)$.

Note that the Koszul complex is exact if $i > n + e + 1$ or $j >> 0$.

- $\beta_{1,1}(X)$: the number of quadrics $Q_i \in I_X$;
- $\beta_{2,1}(X)$ is the number of linear relations of the form $\Sigma L_i Q_i = 0$;
- $\beta_{3,1}(X)$ is the number of linear relations of linear relations;
- $\beta_{1,2}(X)$ is the number of cubic generators of I_X.

By the symmetry of Tor, the graded Betti numbers are also defined via the Koszul exact sequence of the base field K:
$V = K\langle x_0, \cdots, x_{n+e}\rangle$ be the K-vector space in $K[x_0, \dots, x_{n+e}]$.
Then, $\mathrm{Tor}_i^R(R/I_X, K)_{i+j}$ is the homology of the Koszul complex:

$$\wedge^{i+1} V \otimes (R/I_X)_{j-1} \xrightarrow{\varphi_{i+1,j-1}} \wedge^{i} V \otimes (R/I_X)_{j} \xrightarrow{\varphi_{i,j}} \wedge^{i-1} V \otimes (R/I_X)_{j+1},$$

where the map is given by $\varphi_{i,j}(x_{\alpha_1} \wedge x_{\alpha_2} \wedge \cdots \wedge x_{\alpha_i} \otimes m) = \sum_{1\leq\mu\leq i} (-1)^{\mu-1} x_{\alpha_1} \cdots \wedge \hat{x_{\alpha_\mu}} \wedge \ldots \wedge x_{\alpha_i} \otimes (x_{\alpha_\mu} \cdot m)$.

Note that the Koszul complex is exact if $i > n+e+1$ or $j >> 0$.

- $\beta_{1,1}(X)$: the number of quadrics $Q_i \in I_X$;
- $\beta_{2,1}(X)$ is the number of linear relations of the form $\Sigma L_i Q_i = 0$;
- $\beta_{3,1}(X)$ is the number of linear relations of linear relations;
- $\beta_{1,2}(X)$ is the number of cubic generators of I_X.

By the symmetry of Tor, the graded Betti numbers are also defined via the Koszul exact sequence of the base field K:
$V = K\langle x_0, \cdots, x_{n+e}\rangle$ be the K-vector space in $K[x_0, \ldots, x_{n+e}]$.
Then, $\mathrm{Tor}_i^R(R/I_X, K)_{i+j}$ is the homology of the Koszul complex:

$$\wedge^{i+1} V \otimes (R/I_X)_{j-1} \xrightarrow{\varphi_{i+1,j-1}} \wedge^i V \otimes (R/I_X)_j \xrightarrow{\varphi_{i,j}} \wedge^{i-1} V \otimes (R/I_X)_{j+1},$$

where the map is given by $\varphi_{i,j}(x_{\alpha_1} \wedge x_{\alpha_2} \wedge \cdots \wedge x_{\alpha_i} \otimes m) = \sum_{1\leq\mu\leq i} (-1)^{\mu-1} x_{\alpha_1} \cdots \wedge \hat{x_{\alpha_\mu}} \wedge \ldots \wedge x_{\alpha_i} \otimes (x_{\alpha_\mu} \cdot m)$.

Note that the Koszul complex is exact if $i > n+e+1$ or $j >> 0$.

- $\beta_{1,1}(X)$: the number of quadrics $Q_i \in I_X$;
- $\beta_{2,1}(X)$ is the number of linear relations of the form $\Sigma L_i Q_i = 0$;
- $\beta_{3,1}(X)$ is the number of linear relations of linear relations;
- $\beta_{1,2}(X)$ is the number of cubic generators of I_X.

By the symmetry of Tor, the graded Betti numbers are also defined via the Koszul exact sequence of the base field K:
$V = K\langle x_0, \cdots, x_{n+e}\rangle$ be the K-vector space in $K[x_0, \ldots, x_{n+e}]$.
Then, $\mathrm{Tor}_i^R(R/I_X, K)_{i+j}$ is the homology of the Koszul complex:

$$\wedge^{i+1} V \otimes (R/I_X)_{j-1} \xrightarrow{\varphi_{i+1,j-1}} \wedge^{i} V \otimes (R/I_X)_{j} \xrightarrow{\varphi_{i,j}} \wedge^{i-1} V \otimes (R/I_X)_{j+1},$$

where the map is given by $\varphi_{i,j}(x_{\alpha_1} \wedge x_{\alpha_2} \wedge \cdots \wedge x_{\alpha_i} \otimes m) = \sum_{1 \le \mu \le i} (-1)^{\mu-1} x_{\alpha_1} \cdots \wedge \hat{x_{\alpha_\mu}} \wedge \ldots \wedge x_{\alpha_i} \otimes (x_{\alpha_\mu} \cdot m)$.
Note that the Koszul complex is exact if $i > n+e+1$ or $j >> 0$.

- $\beta_{1,1}(X)$: the number of quadrics $Q_i \in I_X$;
- $\beta_{2,1}(X)$ is the number of linear relations of the form $\Sigma L_i Q_i = 0$;
- $\beta_{3,1}(X)$ is the number of linear relations of linear relations;
- $\beta_{1,2}(X)$ is the number of cubic generators of I_X.

By the symmetry of Tor, the graded Betti numbers are also defined via the Koszul exact sequence of the base field K:
$V = K\langle x_0, \cdots, x_{n+e}\rangle$ be the K-vector space in $K[x_0, \ldots, x_{n+e}]$.
Then, $\mathrm{Tor}_i^R(R/I_X, K)_{i+j}$ is the homology of the Koszul complex:

$$\wedge^{i+1} V \otimes (R/I_X)_{j-1} \xrightarrow{\varphi_{i+1,j-1}} \wedge^{i} V \otimes (R/I_X)_{j} \xrightarrow{\varphi_{i,j}} \wedge^{i-1} V \otimes (R/I_X)_{j+1},$$

where the map is given by $\varphi_{i,j}(x_{\alpha_1} \wedge x_{\alpha_2} \wedge \cdots \wedge x_{\alpha_i} \otimes m) = \sum_{1\leq \mu \leq i} (-1)^{\mu-1} x_{\alpha_1} \cdots \wedge \hat{x_{\alpha_\mu}} \wedge \ldots \wedge x_{\alpha_i} \otimes (x_{\alpha_\mu} \cdot m)$.
Note that the Koszul complex is exact if $i > n+e+1$ or $j >> 0$.

- $\beta_{1,1}(X)$: the number of quadrics $Q_i \in I_X$;
- $\beta_{2,1}(X)$ is the number of linear relations of the form $\Sigma L_i Q_i = 0$;
- $\beta_{3,1}(X)$ is the number of linear relations of linear relations;
- $\beta_{1,2}(X)$ is the number of cubic generators of I_X.

By the symmetry of Tor, the graded Betti numbers are also defined via the Koszul exact sequence of the base field K:
$V = K\langle x_0, \cdots, x_{n+e}\rangle$ be the K-vector space in $K[x_0, \ldots, x_{n+e}]$.
Then, $\mathrm{Tor}_i^R(R/I_X, K)_{i+j}$ is the homology of the Koszul complex:

$$\wedge^{i+1} V \otimes (R/I_X)_{j-1} \xrightarrow{\varphi_{i+1,j-1}} \wedge^{i} V \otimes (R/I_X)_{j} \xrightarrow{\varphi_{i,j}} \wedge^{i-1} V \otimes (R/I_X)_{j+1},$$

where the map is given by $\varphi_{i,j}(x_{\alpha_1} \wedge x_{\alpha_2} \wedge \cdots \wedge x_{\alpha_i} \otimes m) = \sum_{1\leq\mu\leq i} (-1)^{\mu-1} x_{\alpha_1} \cdots \wedge \hat{x_{\alpha_\mu}} \wedge \ldots \wedge x_{\alpha_i} \otimes (x_{\alpha_\mu} \cdot m)$.
Note that the Koszul complex is exact if $i > n+e+1$ or $j >> 0$.

- $\beta_{1,1}(X)$: the number of quadrics $Q_i \in I_X$;
- $\beta_{2,1}(X)$ is the number of linear relations of the form $\Sigma L_i Q_i = 0$;
- $\beta_{3,1}(X)$ is the number of linear relations of linear relations;
- $\beta_{1,2}(X)$ is the number of cubic generators of I_X.

By the symmetry of Tor, the graded Betti numbers are also defined via the Koszul exact sequence of the base field K:
$V = K\langle x_0, \cdots, x_{n+e}\rangle$ be the K-vector space in $K[x_0, \ldots, x_{n+e}]$.
Then, $\mathrm{Tor}_i^R(R/I_X, K)_{i+j}$ is the homology of the Koszul complex:

$$\wedge^{i+1} V \otimes (R/I_X)_{j-1} \xrightarrow{\varphi_{i+1,j-1}} \wedge^i V \otimes (R/I_X)_j \xrightarrow{\varphi_{i,j}} \wedge^{i-1} V \otimes (R/I_X)_{j+1},$$

where the map is given by $\varphi_{i,j}(x_{\alpha_1} \wedge x_{\alpha_2} \wedge \cdots \wedge x_{\alpha_i} \otimes m) = \sum_{1\leq\mu\leq i} (-1)^{\mu-1} x_{\alpha_1} \cdots \wedge \hat{x_{\alpha_\mu}} \wedge \ldots \wedge x_{\alpha_i} \otimes (x_{\alpha_\mu} \cdot m)$.
Note that the Koszul complex is exact if $i > n+e+1$ or $j >> 0$.

- $\beta_{1,1}(X)$: the number of quadrics $Q_i \in I_X$;
- $\beta_{2,1}(X)$ is the number of linear relations of the form $\Sigma L_i Q_i = 0$;
- $\beta_{3,1}(X)$ is the number of linear relations of linear relations;
- $\beta_{1,2}(X)$ is the number of cubic generators of I_X.

- Many geometric information on X can be read off from the Betti table (e.g. Green conjecture, gonality conjecture, genus, degree etc).
- One can say that X satisfies property $\mathbf{N}_{2,p}$ (or $\mathbf{N}_p$) if $\beta_{i,j}(X) = 0$ for $1 \leq i \leq p, \quad j \geq 2$.

	0	1	2	3	$\cdots$	p	$p+1$	$\cdots$	$\triangle$
0	1	–	–	–	$\cdots$	–		$\cdots$	–
1	–	$\beta_{1,1}$	$\beta_{2,1}$	$\beta_{3,1}$	$\cdots$	$\beta_{p,1}$	$\beta_{p+1,1}$	$\cdots$	$\beta_{\triangle,1}$
2	–	–	–	–	$\cdots$	–	$\beta_{p+1,2}$	$\cdots$	$\beta_{\triangle,2}$

- We are interested in the Betti numbers in the first linear strand.

	0	1	2	3	$\cdots$	$i-1$	i	$i+1$	$\cdots$	$\triangle$
1	–	$\beta_{1,1}$	$\beta_{2,1}$	$\beta_{3,1}$	$\cdots$	$\beta_{i-1,1}$	$\beta_{i,1}$	$\beta_{i+1,1}$	$\cdots$	$\beta_{\triangle,1}$

- Many geometric information on X can be read off from the Betti table (e.g. Green conjecture, gonality conjecture, genus, degree etc).
- One can say that X satisfies property $\mathbf{N}_{2,p}$ (or $\mathbf{N}_p$) if $\beta_{i,j}(X) = 0$ for $1 \leq i \leq p, \quad j \geq 2$.

	0	1	2	3	$\cdots$	p	$p+1$	$\cdots$	$\triangle$
0	1	–	–	–	$\cdots$	–		$\cdots$	–
1	–	$\beta_{1,1}$	$\beta_{2,1}$	$\beta_{3,1}$	$\cdots$	$\beta_{p,1}$	$\beta_{p+1,1}$	$\cdots$	$\beta_{\triangle,1}$
2	–	–	–	–	$\cdots$	–	$\beta_{p+1,2}$	$\cdots$	$\beta_{\triangle,2}$

- We are interested in the Betti numbers in the first linear strand.

	0	1	2	3	$\cdots$	$i-1$	i	$i+1$	$\cdots$	$\triangle$
1	–	$\beta_{1,1}$	$\beta_{2,1}$	$\beta_{3,1}$	$\cdots$	$\beta_{i-1,1}$	$\beta_{i,1}$	$\beta_{i+1,1}$	$\cdots$	$\beta_{\triangle,1}$

- Many geometric information on X can be read off from the Betti table (e.g. Green conjecture, gonality conjecture, genus, degree etc).
- One can say that X satisfies property $\mathbf{N}_{2,p}$ (or $\mathbf{N}_p$) if $\beta_{i,j}(X) = 0$ for $1 \leq i \leq p, \quad j \geq 2$.

	0	1	2	3	$\cdots$	p	$p+1$	$\cdots$	$\triangle$
0	1	$-$	$-$	$-$	$\cdots$	$-$		$\cdots$	$-$
1	$-$	$\beta_{1,1}$	$\beta_{2,1}$	$\beta_{3,1}$	$\cdots$	$\beta_{p,1}$	$\beta_{p+1,1}$	$\cdots$	$\beta_{\triangle,1}$
2	$-$	$-$	$-$	$-$	$\cdots$	$-$	$\beta_{p+1,2}$	$\cdots$	$\beta_{\triangle,2}$

- We are interested in the Betti numbers in the first linear strand.

	0	1	2	3	$\cdots$	$i-1$	i	$i+1$	$\cdots$	$\triangle$
1	$-$	$\beta_{1,1}$	$\beta_{2,1}$	$\beta_{3,1}$	$\cdots$	$\beta_{i-1,1}$	$\beta_{i,1}$	$\beta_{i+1,1}$	$\cdots$	$\beta_{\triangle,1}$

- Many geometric information on X can be read off from the Betti table (e.g. Green conjecture, gonality conjecture, genus, degree etc).
- One can say that X satisfies property $\mathbf{N}_{2,p}$ (or $\mathbf{N}_p$) if $\beta_{i,j}(X) = 0$ for $1 \leq i \leq p, \quad j \geq 2$.

	0	1	2	3	$\cdots$	p	$p+1$	$\cdots$	$\triangle$
0	1	–	–	–	$\cdots$	–		$\cdots$	–
1	–	$\beta_{1,1}$	$\beta_{2,1}$	$\beta_{3,1}$	$\cdots$	$\beta_{p,1}$	$\beta_{p+1,1}$	$\cdots$	$\beta_{\triangle,1}$
2	–	–	–	–	$\cdots$	–	$\beta_{p+1,2}$	$\cdots$	$\beta_{\triangle,2}$

- We are interested in the Betti numbers in the first linear strand.

	0	1	2	3	$\cdots$	$i-1$	i	$i+1$	$\cdots$	$\triangle$
1	–	$\beta_{1,1}$	$\beta_{2,1}$	$\beta_{3,1}$	$\cdots$	$\beta_{i-1,1}$	$\beta_{i,1}$	$\beta_{i+1,1}$	$\cdots$	$\beta_{\triangle,1}$

- Many geometric information on X can be read off from the Betti table (e.g. Green conjecture, gonality conjecture, genus, degree etc).
- One can say that X satisfies property $\mathbf{N}_{2,p}$ (or $\mathbf{N}_p$) if $\beta_{i,j}(X) = 0$ for $1 \leq i \leq p, \quad j \geq 2$.

	0	1	2	3	$\cdots$	p	$p+1$	$\cdots$	$\triangle$
0	1	–	–	–	$\cdots$	–		$\cdots$	–
1	–	$\beta_{1,1}$	$\beta_{2,1}$	$\beta_{3,1}$	$\cdots$	$\beta_{p,1}$	$\beta_{p+1,1}$	$\cdots$	$\beta_{\triangle,1}$
2	–	–	–	–	$\cdots$	–	$\beta_{p+1,2}$	$\cdots$	$\beta_{\triangle,2}$

- We are interested in the Betti numbers in the first linear strand.

	0	1	2	3	$\cdots$	$i-1$	i	$i+1$	$\cdots$	$\triangle$
1	–	$\beta_{1,1}$	$\beta_{2,1}$	$\beta_{3,1}$	$\cdots$	$\beta_{i-1,1}$	$\beta_{i,1}$	$\beta_{i+1,1}$	$\cdots$	$\beta_{\triangle,1}$

- Many geometric information on X can be read off from the Betti table (e.g. Green conjecture, gonality conjecture, genus, degree etc).
- One can say that X satisfies property $\mathbf{N}_{2,p}$ (or $\mathbf{N}_p$) if $\beta_{i,j}(X) = 0$ for $1 \leq i \leq p, \quad j \geq 2$.

	0	1	2	3	$\cdots$	p	$p+1$	$\cdots$	$\triangle$
0	1	–	–	–	$\cdots$	–		$\cdots$	–
1	–	$\beta_{1,1}$	$\beta_{2,1}$	$\beta_{3,1}$	$\cdots$	$\beta_{p,1}$	$\beta_{p+1,1}$	$\cdots$	$\beta_{\triangle,1}$
2	–	–	–	–	$\cdots$	–	$\beta_{p+1,2}$	$\cdots$	$\beta_{\triangle,2}$

- We are interested in the Betti numbers in the first linear strand.

	0	1	2	3	$\cdots$	$i-1$	i	$i+1$	$\cdots$	$\triangle$
1	–	$\beta_{1,1}$	$\beta_{2,1}$	$\beta_{3,1}$	$\cdots$	$\beta_{i-1,1}$	$\beta_{i,1}$	$\beta_{i+1,1}$	$\cdots$	$\beta_{\triangle,1}$

Natural Philosophy: More quadrics X has, higher linear syzygies can be nonzero and only linear syzygies can happen!

Known Facts:

- [Green, 1984] If $\beta_{p,1} \neq 0$, then $h^0(\mathcal{I}_X(2)) \geq \binom{p+1}{2} = \frac{(p+1)p}{2}$;
- [Han-K, 2012] If X satisfies property $\mathbf{N}_{2,p}$, then $h^0(\mathcal{I}_X(2)) \geq \binom{e+1}{2} - \binom{e+1-p}{2} = \frac{(2e+1-p)p}{2}$.

Recall that the Betti number $\beta_{i,1}$ is defined as the homology of

$$0 \to \wedge^{i+1} V \xrightarrow{\varphi_{i+1,0}} \wedge^i V \otimes V \xrightarrow{\varphi_{i,1}} \wedge^{i-1} V \otimes (R/I_X)_2,$$

and $\beta_{i,1} = \dim_K \mathrm{Tor}_i^R(R_X, K)_{i+1} \leq \dim_K (\wedge^i V \otimes V / \wedge^{i+1} V)$.

Question: What is the sharp bound of $\beta_{i,1}$ for a variety X ?

Natural Philosophy: More quadrics X has, higher linear syzygies can be nonzero and only linear syzygies can happen!

Known Facts:

- [Green, 1984] If $\beta_{p,1} \neq 0$, then $h^0(\mathcal{I}_X(2)) \geq \binom{p+1}{2} = \frac{(p+1)p}{2}$;
- [Han-K, 2012] If X satisfies property $\mathbf{N}_{2,p}$, then $h^0(\mathcal{I}_X(2)) \geq \binom{e+1}{2} - \binom{e+1-p}{2} = \frac{(2e+1-p)p}{2}$.

Recall that the Betti number $\beta_{i,1}$ is defined as the homology of

$$0 \to \wedge^{i+1} V \xrightarrow{\varphi_{i+1,0}} \wedge^i V \otimes V \xrightarrow{\varphi_{i,1}} \wedge^{i-1} V \otimes (R/I_X)_2,$$

and $\beta_{i,1} = \dim_K \operatorname{Tor}_i^R(R_X, K)_{i+1} \leq \dim_K (\wedge^i V \otimes V / \wedge^{i+1} V)$.

Question: What is the sharp bound of $\beta_{i,1}$ for a variety X ?

Natural Philosophy: More quadrics X has, higher linear syzygies can be nonzero and only linear syzygies can happen!

Known Facts:

- [Green, 1984] If $\beta_{p,1} \neq 0$, then $h^0(\mathcal{I}_X(2)) \geq \binom{p+1}{2} = \frac{(p+1)p}{2}$;
- [Han-K, 2012] If X satisfies property $\mathbf{N}_{2,p}$, then $h^0(\mathcal{I}_X(2)) \geq \binom{e+1}{2} - \binom{e+1-p}{2} = \frac{(2e+1-p)p}{2}$.

Recall that the Betti number $\beta_{i,1}$ is defined as the homology of

$$0 \to \wedge^{i+1} V \xrightarrow{\varphi_{i+1,0}} \wedge^i V \otimes V \xrightarrow{\varphi_{i,1}} \wedge^{i-1} V \otimes (R/I_X)_2,$$

and $\beta_{i,1} = \dim_K \operatorname{Tor}_i^R(R_X, K)_{i+1} \leq \dim_K (\wedge^i V \otimes V / \wedge^{i+1} V)$.

Question: What is the sharp bound of $\beta_{i,1}$ for a variety X ?

Natural Philosophy: More quadrics X has, higher linear syzygies can be nonzero and only linear syzygies can happen!

Known Facts:

- [Green, 1984] If $\beta_{p,1} \neq 0$, then $h^0(\mathcal{I}_X(2)) \geq \binom{p+1}{2} = \frac{(p+1)p}{2}$;
- [Han-K, 2012] If X satisfies property $\mathbf{N}_{2,p}$, then $h^0(\mathcal{I}_X(2)) \geq \binom{e+1}{2} - \binom{e+1-p}{2} = \frac{(2e+1-p)p}{2}$.

Recall that the Betti number $\beta_{i,1}$ is defined as the homology of

$$0 \to \wedge^{i+1} V \xrightarrow{\varphi_{i+1,0}} \wedge^i V \otimes V \xrightarrow{\varphi_{i,1}} \wedge^{i-1} V \otimes (R/I_X)_2,$$

and $\beta_{i,1} = \dim_K \operatorname{Tor}_i^R(R_X, K)_{i+1} \leq \dim_K (\wedge^i V \otimes V / \wedge^{i+1} V)$.

Question: What is the sharp bound of $\beta_{i,1}$ for a variety X ?

Natural Philosophy: More quadrics X has, higher linear syzygies can be nonzero and only linear syzygies can happen!

Known Facts:

- [Green, 1984] If $\beta_{p,1} \neq 0$, then $h^0(\mathcal{I}_X(2)) \geq \binom{p+1}{2} = \frac{(p+1)p}{2}$;
- [Han-K, 2012] If X satisfies property $\mathbf{N}_{2,p}$, then $h^0(\mathcal{I}_X(2)) \geq \binom{e+1}{2} - \binom{e+1-p}{2} = \frac{(2e+1-p)p}{2}$.

Recall that the Betti number $\beta_{i,1}$ is defined as the homology of

$$0 \to \wedge^{i+1} V \xrightarrow{\varphi_{i+1,0}} \wedge^{i} V \otimes V \xrightarrow{\varphi_{i,1}} \wedge^{i-1} V \otimes (R/I_X)_2,$$

and $\beta_{i,1} = \dim_K \mathrm{Tor}_i^R(R_X, K)_{i+1} \leq \dim_K \left(\wedge^i V \otimes V / \wedge^{i+1} V\right)$.

Question: What is the sharp bound of $\beta_{i,1}$ for a variety X ?

Natural Philosophy: More quadrics X has, higher linear syzygies can be nonzero and only linear syzygies can happen!

Known Facts:

- [Green, 1984] If $\beta_{p,1} \neq 0$, then $h^0(\mathcal{I}_X(2)) \geq \binom{p+1}{2} = \frac{(p+1)p}{2}$;
- [Han-K, 2012] If X satisfies property $\mathbf{N}_{2,p}$, then $h^0(\mathcal{I}_X(2)) \geq \binom{e+1}{2} - \binom{e+1-p}{2} = \frac{(2e+1-p)p}{2}$.

Recall that the Betti number $\beta_{i,1}$ is defined as the homology of

$$0 \to \wedge^{i+1} V \xrightarrow{\varphi_{i+1,0}} \wedge^i V \otimes V \xrightarrow{\varphi_{i,1}} \wedge^{i-1} V \otimes (R/I_X)_2,$$

and $\beta_{i,1} = \dim_K \mathrm{Tor}_i^R(R_X, K)_{i+1} \leq \dim_K (\wedge^i V \otimes V / \wedge^{i+1} V)$.

Question: What is the sharp bound of $\beta_{i,1}$ for a variety X ?

Natural Philosophy: More quadrics X has, higher linear syzygies can be nonzero and only linear syzygies can happen!

Known Facts:

- [Green, 1984] If $\beta_{p,1} \neq 0$, then $h^0(\mathcal{I}_X(2)) \geq \binom{p+1}{2} = \frac{(p+1)p}{2}$;
- [Han-K, 2012] If X satisfies property $\mathbf{N}_{2,p}$, then $h^0(\mathcal{I}_X(2)) \geq \binom{e+1}{2} - \binom{e+1-p}{2} = \frac{(2e+1-p)p}{2}$.

Recall that the Betti number $\beta_{i,1}$ is defined as the homology of

$$0 \to \wedge^{i+1} V \xrightarrow{\varphi_{i+1,0}} \wedge^i V \otimes V \xrightarrow{\varphi_{i,1}} \wedge^{i-1} V \otimes (R/I_X)_2,$$

and $\beta_{i,1} = \dim_K \mathrm{Tor}_i^R(R_X, K)_{i+1} \leq \dim_K (\wedge^i V \otimes V / \wedge^{i+1} V)$.

Question: What is the sharp bound of $\beta_{i,1}$ for a variety X ?

Classical Question: How many possible quadric hypersurfaces containing $X \subset \mathbb{P}(V)$, i.e. $\exists$ an upper bound of $\beta_{1,1}$?

The simplest examples of curves

- A rational normal curve $C \subset \mathbb{P}^{1+e}$:
$0 \to H^0(\mathcal{I}_C(2)) \to H^0(\mathcal{O}_{\mathbb{P}^{e+1}}(2)) \to H^0(\mathcal{O}_C(2)) \to 0$ and by R-R,
$h^0(\mathcal{I}_C(2)) = h^0(\mathcal{O}_{\mathbb{P}^{e+1}}(2)) - h^0(\mathcal{O}_C(2)) = \binom{e+3}{2} - (2(e+1)+1) = \binom{e+1}{2}$.

- An elliptic normal curve $C \subset \mathbb{P}^{1+e}$: Similarly, we have
$h^0(\mathcal{I}_C(2)) = \binom{e+3}{2} - (2(e+2)+1-g(C)) = \binom{e+1}{2} - 1$.

□ **Castelnuovo**(1889)
$h^0(\mathcal{I}_{X/\mathbb{P}^{n+e}}(2)) \leq \binom{e+1}{2}$ and " $=$ " holds iff X is a variety of minimal degree, i.e. $\deg(X) = e+1$.

Classical Question: How many possible quadric hypersurfaces containing $X \subset \mathbb{P}(V)$, i.e. $\exists$ an upper bound of $\beta_{1,1}$?

The simplest examples of curves

- A rational normal curve $C \subset \mathbb{P}^{1+e}$:
$0 \to H^0(\mathcal{I}_C(2)) \to H^0(\mathcal{O}_{\mathbb{P}^{e+1}}(2)) \to H^0(\mathcal{O}_C(2)) \to 0$ and by R-R,
$h^0(\mathcal{I}_C(2)) = h^0(\mathcal{O}_{\mathbb{P}^{e+1}}(2)) - h^0(\mathcal{O}_C(2)) = \binom{e+3}{2} - (2(e+1)+1) = \binom{e+1}{2}$.

- An elliptic normal curve $C \subset \mathbb{P}^{1+e}$: Similarly, we have
$h^0(\mathcal{I}_C(2)) = \binom{e+3}{2} - (2(e+2)+1-g(C)) = \binom{e+1}{2} - 1$.

□ **Castelnuovo**(1889)
$h^0(\mathcal{I}_{X/\mathbb{P}^{n+e}}(2)) \leq \binom{e+1}{2}$ and " $=$ " holds iff X is a variety of minimal degree, i.e. $\deg(X) = e+1$.

Classical Question: How many possible quadric hypersurfaces containing $X \subset \mathbb{P}(V)$, i.e. $\exists$ an upper bound of $\beta_{1,1}$?

The simplest examples of curves

- A rational normal curve $C \subset \mathbb{P}^{1+e}$:
$0 \to H^0(\mathcal{I}_C(2)) \to H^0(\mathcal{O}_{\mathbb{P}^{e+1}}(2)) \to H^0(\mathcal{O}_C(2)) \to 0$ and by R-R,
$h^0(\mathcal{I}_C(2)) = h^0(\mathcal{O}_{\mathbb{P}^{e+1}}(2)) - h^0(\mathcal{O}_C(2)) = \binom{e+3}{2} - (2(e+1)+1) = \binom{e+1}{2}$.

- An elliptic normal curve $C \subset \mathbb{P}^{1+e}$: Similarly, we have
$h^0(\mathcal{I}_C(2)) = \binom{e+3}{2} - (2(e+2)+1-g(C)) = \binom{e+1}{2} - 1$.

□ **Castelnuovo**(1889)
$h^0(\mathcal{I}_{X/\mathbb{P}^{n+e}}(2)) \leq \binom{e+1}{2}$ and " $=$ " holds iff X is a variety of minimal degree, i.e. $\deg(X) = e+1$.

Classical Question: How many possible quadric hypersurfaces containing $X \subset \mathbb{P}(V)$, i.e. $\exists$ an upper bound of $\beta_{1,1}$?

The simplest examples of curves

- A rational normal curve $C \subset \mathbb{P}^{1+e}$:
$0 \to H^0(\mathcal{I}_C(2)) \to H^0(\mathcal{O}_{\mathbb{P}^{e+1}}(2)) \to H^0(\mathcal{O}_C(2)) \to 0$ and by R-R,
$h^0(\mathcal{I}_C(2)) = h^0(\mathcal{O}_{\mathbb{P}^{e+1}}(2)) - h^0(\mathcal{O}_C(2)) = \binom{e+3}{2} - (2(e+1)+1) = \binom{e+1}{2}$.

- An elliptic normal curve $C \subset \mathbb{P}^{1+e}$: Similarly, we have
$h^0(\mathcal{I}_C(2)) = \binom{e+3}{2} - (2(e+2)+1-g(C)) = \binom{e+1}{2} - 1$.

☐ **Castelnuovo**(1889)
$h^0(\mathcal{I}_{X/\mathbb{P}^{n+e}}(2)) \leq \binom{e+1}{2}$ and " $=$ " holds iff X is a variety of minimal degree, i.e. $\deg(X) = e+1$.

Classical Question: How many possible quadric hypersurfaces containing $X \subset \mathbb{P}(V)$, i.e. $\exists$ an upper bound of $\beta_{1,1}$?

The simplest examples of curves

- A rational normal curve $C \subset \mathbb{P}^{1+e}$:
$0 \to H^0(\mathcal{I}_C(2)) \to H^0(\mathcal{O}_{\mathbb{P}^{e+1}}(2)) \to H^0(\mathcal{O}_C(2)) \to 0$ and by R-R,
$h^0(\mathcal{I}_C(2)) = h^0(\mathcal{O}_{\mathbb{P}^{e+1}}(2)) - h^0(\mathcal{O}_C(2)) = \binom{e+3}{2} - (2(e+1)+1) = \binom{e+1}{2}$.

- An elliptic normal curve $C \subset \mathbb{P}^{1+e}$: Similarly, we have
$h^0(\mathcal{I}_C(2)) = \binom{e+3}{2} - (2(e+2)+1-g(C)) = \binom{e+1}{2} - 1$.

□ **Castelnuovo**(1889)
$h^0(\mathcal{I}_{X/\mathbb{P}^{n+e}}(2)) \leq \binom{e+1}{2}$ and " $=$ " holds iff X is a variety of minimal degree, i.e. $\deg(X) = e+1$.

Classical Question: How many possible quadric hypersurfaces containing $X \subset \mathbb{P}(V)$, i.e. $\exists$ an upper bound of $\beta_{1,1}$?

The simplest examples of curves

- A rational normal curve $C \subset \mathbb{P}^{1+e}$:
$0 \to H^0(\mathcal{I}_C(2)) \to H^0(\mathcal{O}_{\mathbb{P}^{e+1}}(2)) \to H^0(\mathcal{O}_C(2)) \to 0$ and by R-R,
$h^0(\mathcal{I}_C(2)) = h^0(\mathcal{O}_{\mathbb{P}^{e+1}}(2)) - h^0(\mathcal{O}_C(2)) = \binom{e+3}{2} - (2(e+1)+1) = \binom{e+1}{2}$.

- An elliptic normal curve $C \subset \mathbb{P}^{1+e}$: Similarly, we have
$h^0(\mathcal{I}_C(2)) = \binom{e+3}{2} - (2(e+2)+1-g(C)) = \binom{e+1}{2} - 1$.

□ **Castelnuovo**(1889)
$h^0(\mathcal{I}_{X/\mathbb{P}^{n+e}}(2)) \leq \binom{e+1}{2}$ and " $=$ " holds iff X is a variety of minimal degree, i.e. $\deg(X) = e+1$.

Upper bound of the number of quadrics

Castelnuovo's simple proof.

$\Gamma = X \cap \mathbb{P}^e$ is a set of d-points in general position for general $\mathbb{P}^e$.
Since $d \geq e+1$, take a subset $\Gamma' = \{p_1, p_1, \ldots, p_{e+1}\} \subset \Gamma \subset \mathbb{P}^e$.
$h^0(\mathcal{I}_X(2)) \leq h^0(\mathcal{I}_\Gamma(2)) \leq h^0(\mathcal{I}_{\Gamma'}(2)) = \binom{e+2}{2} - (e+1) = \binom{e+1}{2}$.

□ Further results in this method.

- (Fano, 1894) Unless X is VMD, $h^0(\mathcal{I}_X(2)) \leq \binom{e+1}{2} - 1$ and " $=$ " holds iff X is a del Pezzo variety (i.e. arithmetically Cohen-Macaulay and $\deg(X) = e+2$).
- Note that the curve sections of a VMD and a del Pezzo variety are the rational normal curve, smooth elliptic normal curve and a rational nodal curve.

Upper bound of the number of quadrics

Castelnuovo's simple proof.

$\Gamma = X \cap \mathbb{P}^e$ is a set of d-points in general position for general $\mathbb{P}^e$.
Since $d \geq e+1$, take a subset $\Gamma' = \{p_1, p_1, \ldots, p_{e+1}\} \subset \Gamma \subset \mathbb{P}^e$.
$h^0(\mathcal{I}_X(2)) \leq h^0(\mathcal{I}_\Gamma(2)) \leq h^0(\mathcal{I}_{\Gamma'}(2)) = \binom{e+2}{2} - (e+1) = \binom{e+1}{2}$.

☐ Further results in this method.

- (Fano, 1894) Unless X is VMD, $h^0(\mathcal{I}_X(2)) \leq \binom{e+1}{2} - 1$ and " $=$ " holds iff X is a del Pezzo variety (i.e. arithmetically Cohen-Macaulay and $\deg(X) = e+2$).
- Note that the curve sections of a VMD and a del Pezzo variety are the rational normal curve, smooth elliptic normal curve and a rational nodal curve.

Upper bound of the number of quadrics

Castelnuovo's simple proof.

$\Gamma = X \cap \mathbb{P}^e$ is a set of d-points in general position for general $\mathbb{P}^e$.
Since $d \geq e+1$, take a subset $\Gamma' = \{p_1, p_1, \ldots, p_{e+1}\} \subset \Gamma \subset \mathbb{P}^e$.
$h^0(\mathcal{I}_X(2)) \leq h^0(\mathcal{I}_\Gamma(2)) \leq h^0(\mathcal{I}_{\Gamma'}(2)) = \binom{e+2}{2} - (e+1) = \binom{e+1}{2}$.

□ Further results in this method.

- (Fano, 1894) Unless X is VMD, $h^0(\mathcal{I}_X(2)) \leq \binom{e+1}{2} - 1$ and " $=$ " holds iff X is a del Pezzo variety (i.e. arithmetically Cohen-Macaulay and $\deg(X) = e+2$).
- Note that the curve sections of a VMD and a del Pezzo variety are the rational normal curve, smooth elliptic normal curve and a rational nodal curve.

Upper bound of the number of quadrics

Castelnuovo's simple proof.

$\Gamma = X \cap \mathbb{P}^e$ is a set of d-points in general position for general $\mathbb{P}^e$.
Since $d \geq e + 1$, take a subset $\Gamma' = \{p_1, p_1, \ldots, p_{e+1}\} \subset \Gamma \subset \mathbb{P}^e$.
$h^0(\mathcal{I}_X(2)) \leq h^0(\mathcal{I}_\Gamma(2)) \leq h^0(\mathcal{I}_{\Gamma'}(2)) = \binom{e+2}{2} - (e+1) = \binom{e+1}{2}$.

- Further results in this method.
  - (Fano, 1894) Unless X is VMD, $h^0(\mathcal{I}_X(2)) \leq \binom{e+1}{2} - 1$ and " $=$ " holds iff X is a del Pezzo variety (i.e. arithmetically Cohen-Macaulay and $\deg(X) = e + 2$).
  - Note that the curve sections of a VMD and a del Pezzo variety are the rational normal curve, smooth elliptic normal curve and a rational nodal curve.

Upper bound of the number of quadrics

Castelnuovo's simple proof.

$\Gamma = X \cap \mathbb{P}^e$ is a set of d-points in general position for general $\mathbb{P}^e$.
Since $d \geq e+1$, take a subset $\Gamma' = \{p_1, p_1, \ldots, p_{e+1}\} \subset \Gamma \subset \mathbb{P}^e$.
$h^0(\mathcal{I}_X(2)) \leq h^0(\mathcal{I}_\Gamma(2)) \leq h^0(\mathcal{I}_{\Gamma'}(2)) = \binom{e+2}{2} - (e+1) = \binom{e+1}{2}$.

□ Further results in this method.

- (Fano, 1894) Unless X is VMD, $h^0(\mathcal{I}_X(2)) \leq \binom{e+1}{2} - 1$ and " $=$ " holds iff X is a del Pezzo variety (i.e. arithmetically Cohen-Macaulay and $\deg(X) = e+2$).
- Note that the curve sections of a VMD and a del Pezzo variety are the rational normal curve, smooth elliptic normal curve and a rational nodal curve.

Upper bound of the number of quadrics

Castelnuovo's simple proof.

$\Gamma = X \cap \mathbb{P}^e$ is a set of d-points in general position for general $\mathbb{P}^e$.
Since $d \geq e+1$, take a subset $\Gamma' = \{p_1, p_1, \dots, p_{e+1}\} \subset \Gamma \subset \mathbb{P}^e$.
$h^0(\mathcal{I}_X(2)) \leq h^0(\mathcal{I}_\Gamma(2)) \leq h^0(\mathcal{I}_{\Gamma'}(2)) = \binom{e+2}{2} - (e+1) = \binom{e+1}{2}$.

□ Further results in this method.

- (Fano, 1894) Unless X is VMD, $h^0(\mathcal{I}_X(2)) \leq \binom{e+1}{2} - 1$ and " $=$ " holds iff X is a del Pezzo variety (i.e. arithmetically Cohen-Macaulay and $\deg(X) = e+2$).
- Note that the curve sections of a VMD and a del Pezzo variety are the rational normal curve, smooth elliptic normal curve and a rational nodal curve.

Upper bound of the number of quadrics

Castelnuovo's simple proof.

$\Gamma = X \cap \mathbb{P}^e$ is a set of d-points in general position for general $\mathbb{P}^e$.
Since $d \geq e+1$, take a subset $\Gamma' = \{p_1, p_1, \dots, p_{e+1}\} \subset \Gamma \subset \mathbb{P}^e$.
$h^0(\mathcal{I}_X(2)) \leq h^0(\mathcal{I}_\Gamma(2)) \leq h^0(\mathcal{I}_{\Gamma'}(2)) = \binom{e+2}{2} - (e+1) = \binom{e+1}{2}$.

□ Further results in this method.

- (Fano, 1894) Unless X is VMD, $h^0(\mathcal{I}_X(2)) \leq \binom{e+1}{2} - 1$ and " $=$ " holds iff X is a del Pezzo variety (i.e. arithmetically Cohen-Macaulay and $\deg(X) = e+2$).
- Note that the curve sections of a VMD and a del Pezzo variety are the rational normal curve, smooth elliptic normal curve and a rational nodal curve.

Our observation using inner projections.

Consider $\pi_q : X \dashrightarrow X_q \subset \mathbb{P}^{n+e-1}$ from a smooth point $q \in X$:

$$\begin{array}{ccc} \mathrm{Bl}_q(X) \simeq \tilde{X} & & \\ \downarrow \sigma & \searrow \tilde{\pi_q} & \\ X \subset \mathbb{P}^{n+e} & \dashrightarrow & X_q = \overline{\pi_q(X \setminus \{q\})} \subset \mathbb{P}^{n+e-1}. \end{array}$$

Lemma

$X^n \subset \mathbb{P}^{n+e}$: irreducible and reduced (not necessarily smooth). Then, we have the following:

1. $h^0(\mathcal{I}_{X_q}(2)) \leq h^0(\mathbb{P}^{n+e}, \mathcal{I}_X(2)) \leq h^0(\mathcal{I}_{X_q}(2)) + e$.
2. In the second inequality, " $=$ " holds iff X is a local complete intersection of quadrics.

Our observation using inner projections.

Consider $\pi_q : X \dashrightarrow X_q \subset \mathbb{P}^{n+e-1}$ from a smooth point $q \in X$:

$$\begin{array}{ccc} \mathrm{Bl}_q(X) \simeq \tilde{X} & & \\ \downarrow \sigma & \searrow^{\tilde{\pi_q}} & \\ X \subset \mathbb{P}^{n+e} & \dashrightarrow & X_q = \overline{\pi_q(X \setminus \{q\})} \subset \mathbb{P}^{n+e-1}. \end{array}$$

Lemma

$X^n \subset \mathbb{P}^{n+e}$: irreducible and reduced (not necessarily smooth). Then, we have the following:

1. $h^0(\mathcal{I}_{X_q}(2)) \leq h^0(\mathbb{P}^{n+e}, \mathcal{I}_X(2)) \leq h^0(\mathcal{I}_{X_q}(2)) + e$.
2. In the second inequality, " $=$ " holds iff X is a local complete intersection of quadrics.

Our observation using inner projections.

Consider $\pi_q : X \dashrightarrow X_q \subset \mathbb{P}^{n+e-1}$ from a smooth point $q \in X$:

$$\begin{array}{ccc} \mathrm{Bl}_q(X) \simeq \tilde{X} & & \\ \downarrow \sigma & \searrow^{\tilde{\pi_q}} & \\ X \subset \mathbb{P}^{n+e} & \dashrightarrow & X_q = \overline{\pi_q(X \setminus \{q\})} \subset \mathbb{P}^{n+e-1}. \end{array}$$

Lemma

$X^n \subset \mathbb{P}^{n+e}$: irreducible and reduced (not necessarily smooth). Then, we have the following:

1. $h^0(\mathcal{I}_{X_q}(2)) \leq h^0(\mathbb{P}^{n+e}, \mathcal{I}_X(2)) \leq h^0(\mathcal{I}_{X_q}(2)) + e$.
2. In the second inequality, " $=$ " holds iff X is a local complete intersection of quadrics.

Our observation using inner projections.

Consider $\pi_q : X \dashrightarrow X_q \subset \mathbb{P}^{n+e-1}$ from a smooth point $q \in X$:

$$\begin{array}{ccc} \mathrm{Bl}_q(X) \simeq \tilde{X} & & \\ \downarrow \sigma & \searrow \tilde{\pi_q} & \\ X \subset \mathbb{P}^{n+e} & \dashrightarrow & X_q = \overline{\pi_q(X \setminus \{q\})} \subset \mathbb{P}^{n+e-1}. \end{array}$$

Lemma

$X^n \subset \mathbb{P}^{n+e}$: irreducible and reduced (not necessarily smooth). Then, we have the following:

1. $h^0(\mathcal{I}_{X_q}(2)) \leq h^0(\mathbb{P}^{n+e}, \mathcal{I}_X(2)) \leq h^0(\mathcal{I}_{X_q}(2)) + e$.
2. In the second inequality, " $=$ " holds iff X is a local complete intersection of quadrics.

Our observation using inner projections.

Consider $\pi_q : X \dashrightarrow X_q \subset \mathbb{P}^{n+e-1}$ from a smooth point $q \in X$:

$$\begin{array}{ccc} \mathrm{Bl}_q(X) \simeq \tilde{X} & & \\ \downarrow \sigma & \searrow^{\tilde{\pi_q}} & \\ X \subset \mathbb{P}^{n+e} & \dashrightarrow & X_q = \overline{\pi_q(X \setminus \{q\})} \subset \mathbb{P}^{n+e-1}. \end{array}$$

Lemma

$X^n \subset \mathbb{P}^{n+e}$: irreducible and reduced (not necessarily smooth). Then, we have the following:

1. $h^0(\mathcal{I}_{X_q}(2)) \leq h^0(\mathbb{P}^{n+e}, \mathcal{I}_X(2)) \leq h^0(\mathcal{I}_{X_q}(2)) + e$.
2. In the second inequality, " $=$ " holds iff X is a local complete intersection of quadrics.

Our observation using inner projections.

Consider $\pi_q : X \dashrightarrow X_q \subset \mathbb{P}^{n+e-1}$ from a smooth point $q \in X$:

$$\begin{array}{ccc} \mathrm{Bl}_q(X) \simeq \tilde{X} & & \\ \downarrow \sigma & \searrow^{\tilde{\pi_q}} & \\ X \subset \mathbb{P}^{n+e} & \dashrightarrow & X_q = \overline{\pi_q(X \setminus \{q\})} \subset \mathbb{P}^{n+e-1}. \end{array}$$

Lemma

$X^n \subset \mathbb{P}^{n+e}$: irreducible and reduced (not necessarily smooth). Then, we have the following:

1. $h^0(\mathcal{I}_{X_q}(2)) \leq h^0(\mathbb{P}^{n+e}, \mathcal{I}_X(2)) \leq h^0(\mathcal{I}_{X_q}(2)) + e$.
2. In the second inequality, " $=$ " holds iff X is a local complete intersection of quadrics.

[simple proof] Note that $h^0(\mathcal{I}_{X_q}(2)) \leq h^0(\mathbb{P}^{n+e}, \mathcal{I}_X(2))$ is trivial. For the second inequality, we assume a smooth point $q = (1, 0, \cdots, 0) \in X$.

A quadratic form vanishing on X has no x_0^2-term and can be written by the (Gauss) elimination as follows:

$$x_0L_1 + Q_1, \cdots, x_0L_t + Q_t, \tilde{Q}_1, \cdots, \tilde{Q}_\mu$$

where L_i is a linear form and $Q_i, \tilde{Q}_j$ are quadrics in $k[x_1, x_2, \cdots, x_{n+e}]$. Note that $t \leq e$ and $\mu = h^0(\mathcal{I}_{X_q}(2))$. Therefore, by successive inner projections from smooth points up to a hypersurface Z in $\mathbb{P}^{n+1}$,

$$h^0(\mathcal{I}_X(2)) \leq h^0(\mathcal{I}_{X_q}(2)) + e \leq 1 + 2 + \cdots + e = \binom{e+1}{2}.$$

Furthermore, $h^0(\mathcal{I}_X(2)) \leq \binom{e+1}{2}$ iff a hypersurface Z in $\mathbb{P}^{n+1}$ is quadric.

[simple proof] Note that $h^0(\mathcal{I}_{X_q}(2)) \leq h^0(\mathbb{P}^{n+e}, \mathcal{I}_X(2))$ is trivial.
For the second inequality, we assume a smooth point $q = (1, 0, \cdots, 0) \in X$.
A quadratic form vanishing on X has no x_0^2-term and can be written by the (Gauss) elimination as follows:

$$x_0 L_1 + Q_1, \cdots, x_0 L_t + Q_t, \tilde{Q}_1, \cdots, \tilde{Q}_\mu$$

where L_i is a linear form and $Q_i, \tilde{Q}_j$ are quadrics in $k[x_1, x_2, \cdots, x_{n+e}]$. Note that $t \leq e$ and $\mu = h^0(\mathcal{I}_{X_q}(2))$. Therefore, by successive inner projections from smooth points up to a hypersurface Z in $\mathbb{P}^{n+1}$,

$$h^0(\mathcal{I}_X(2)) \leq h^0(\mathcal{I}_{X_q}(2)) + e \leq 1 + 2 + \cdots + e = \binom{e+1}{2}.$$

Furthermore, $h^0(\mathcal{I}_X(2)) \leq \binom{e+1}{2}$ iff a hypersurface Z in $\mathbb{P}^{n+1}$ is quadric.

[simple proof] Note that $h^0(\mathcal{I}_{X_q}(2)) \leq h^0(\mathbb{P}^{n+e}, \mathcal{I}_X(2))$ is trivial.
For the second inequality, we assume a smooth point $q = (1, 0, \cdots, 0) \in X$.
A quadratic form vanishing on X has no x_0^2-term and can be written by the (Gauss) elimination as follows:

$$x_0 L_1 + Q_1, \cdots, x_0 L_t + Q_t, \tilde{Q}_1, \cdots, \tilde{Q}_\mu$$

where L_i is a linear form and $Q_i, \tilde{Q}_j$ are quadrics in $k[x_1, x_2, \cdots, x_{n+e}]$.
Note that $t \leq e$ and $\mu = h^0(\mathcal{I}_{X_q}(2))$. Therefore, by successive inner projections from smooth points up to a hypersurface Z in $\mathbb{P}^{n+1}$,

$$h^0(\mathcal{I}_X(2)) \leq h^0(\mathcal{I}_{X_q}(2)) + e \leq 1 + 2 + \cdots + e = \binom{e+1}{2}.$$

Furthermore, $h^0(\mathcal{I}_X(2)) \leq \binom{e+1}{2}$ iff a hypersurface Z in $\mathbb{P}^{n+1}$ is quadric.

[simple proof] Note that $h^0(\mathcal{I}_{X_q}(2)) \leq h^0(\mathbb{P}^{n+e}, \mathcal{I}_X(2))$ is trivial.
For the second inequality, we assume a smooth point $q = (1, 0, \cdots, 0) \in X$.
A quadratic form vanishing on X has no x_0^2-term and can be written by the (Gauss) elimination as follows:

$$x_0 L_1 + Q_1, \cdots, x_0 L_t + Q_t, \tilde{Q}_1, \cdots, \tilde{Q}_\mu$$

where L_i is a linear form and $Q_i, \tilde{Q}_j$ are quadrics in $k[x_1, x_2, \cdots, x_{n+e}]$.
Note that $t \leq e$ and $\mu = h^0(\mathcal{I}_{X_q}(2))$. Therefore, by successive inner projections from smooth points up to a hypersurface Z in $\mathbb{P}^{n+1}$,

$$h^0(\mathcal{I}_X(2)) \leq h^0(\mathcal{I}_{X_q}(2)) + e \leq 1 + 2 + \cdots + e = \binom{e+1}{2}.$$

Furthermore, $h^0(\mathcal{I}_X(2)) \leq \binom{e+1}{2}$ iff a hypersurface Z in $\mathbb{P}^{n+1}$ is quadric.

[simple proof] Note that $h^0(\mathcal{I}_{X_q}(2)) \leq h^0(\mathbb{P}^{n+e}, \mathcal{I}_X(2))$ is trivial.
For the second inequality, we assume a smooth point $q = (1, 0, \cdots, 0) \in X$.
A quadratic form vanishing on X has no x_0^2-term and can be written by the (Gauss) elimination as follows:

$$x_0 L_1 + Q_1, \cdots, x_0 L_t + Q_t, \tilde{Q}_1, \cdots, \tilde{Q}_\mu$$

where L_i is a linear form and $Q_i, \tilde{Q}_j$ are quadrics in $k[x_1, x_2, \cdots, x_{n+e}]$.
Note that $t \leq e$ and $\mu = h^0(\mathcal{I}_{X_q}(2))$. Therefore, by successive inner projections from smooth points up to a hypersurface Z in $\mathbb{P}^{n+1}$,

$$h^0(\mathcal{I}_X(2)) \leq h^0(\mathcal{I}_{X_q}(2)) + e \leq 1 + 2 + \cdots + e = \binom{e+1}{2}.$$

Furthermore, $h^0(\mathcal{I}_X(2)) \leq \binom{e+1}{2}$ iff a hypersurface Z in $\mathbb{P}^{n+1}$ is quadric.

More general inequality (Han-K, 2015)

- $\beta_{i,1}(X) \leq \beta_{i,1}(X_q) + \beta_{i-1,1}(X_q) + \binom{e}{i}, \quad i \geq 1.$
- The equality holds for $i \leq p$ if X satisfies property $\mathbf{N}_{2,\,p}$.

Using the above inequality, we have the following:

Theorem $X \subset \mathbb{P}^{n+e}$: irreducible and reduced (not necessarily smooth).

$$\beta_{p,1}(X) \leq p\binom{e+1}{p+1}, \quad p \geq 1$$

and the following are equivalent:

(a) X is a variety of minimal degree;

(b) one $\beta_{p,1}(X)$ achieves the maximum for some $p \geq 1$;

(c) X is 2-regular ACM.

More general inequality (Han-K, 2015)

- $\beta_{i,1}(X) \leq \beta_{i,1}(X_q) + \beta_{i-1,1}(X_q) + \binom{e}{i}, \quad i \geq 1.$
- The equality holds for $i \leq p$ if X satisfies property $\mathbf{N}_{2,\,p}$.

Using the above inequality, we have the following:

Theorem $X \subset \mathbb{P}^{n+e}$: irreducible and reduced (not necessarily smooth).

$$\beta_{p,1}(X) \leq p\binom{e+1}{p+1}, \quad p \geq 1$$

and the following are equivalent:

(a) X is a variety of minimal degree;

(b) one $\beta_{p,1}(X)$ achieves the maximum for some $p \geq 1$;

(c) X is 2-regular ACM.

More general inequality (Han-K, 2015)

- $\beta_{i,1}(X) \leq \beta_{i,1}(X_q) + \beta_{i-1,1}(X_q) + \binom{e}{i}, \quad i \geq 1.$
- The equality holds for $i \leq p$ if X satisfies property $\mathbf{N}_{2,\,p}$.

Using the above inequality, we have the following:

Theorem $X \subset \mathbb{P}^{n+e}$: irreducible and reduced (not necessarily smooth).

$$\beta_{p,1}(X) \leq p\binom{e+1}{p+1}, \quad p \geq 1$$

and the following are equivalent:

(a) X is a variety of minimal degree;

(b) one $\beta_{p,1}(X)$ achieves the maximum for some $p \geq 1$;

(c) X is 2-regular ACM.

More general inequality (Han-K, 2015)

- $\beta_{i,1}(X) \leq \beta_{i,1}(X_q) + \beta_{i-1,1}(X_q) + \binom{e}{i}, \quad i \geq 1.$
- The equality holds for $i \leq p$ if X satisfies property $\mathbf{N}_{2,\,p}$.

Using the above inequality, we have the following:

Theorem $X \subset \mathbb{P}^{n+e}$: irreducible and reduced (not necessarily smooth).

$$\beta_{p,1}(X) \leq p\binom{e+1}{p+1}, \quad p \geq 1$$

and the following are equivalent:

(a) X is a variety of minimal degree;

(b) one $\beta_{p,1}(X)$ achieves the maximum for some $p \geq 1$;

(c) X is 2-regular ACM.

More general inequality (Han-K, 2015)

- $\beta_{i,1}(X) \leq \beta_{i,1}(X_q) + \beta_{i-1,1}(X_q) + \binom{e}{i}, \quad i \geq 1.$
- The equality holds for $i \leq p$ if X satisfies property $\mathbf{N}_{2,\,p}$.

Using the above inequality, we have the following:

Theorem $X \subset \mathbb{P}^{n+e}$: irreducible and reduced (not necessarily smooth).

$$\beta_{p,1}(X) \leq p\binom{e+1}{p+1}, \quad p \geq 1$$

and the following are equivalent:

(a) X is a variety of minimal degree;

(b) one $\beta_{p,1}(X)$ achieves the maximum for some $p \geq 1$;

(c) X is 2-regular ACM.

We also characterize Fano varieties as follows:

Theorem [Han-K, 2015]

Unless X is a variety of minimal degree, then we have

$$\beta_{p,1}(X) \leq p\binom{e+1}{p+1} - \binom{e}{p-1} \quad \textit{for all } p \leq e \,,$$

and the following are equivalent:

(a) X is a del Pezzo variety;

(c) one $\beta_{p,1}(X)$ achieves the maximum for some $1 \leq p \leq e-1$;

(d) all the $\beta_{p,1}(X)$ *for* $1 \leq p \leq e-1$ achieve the maximum;

We also characterize Fano varieties as follows:

Theorem [Han-K, 2015]

Unless X is a variety of minimal degree, then we have

$$\beta_{p,1}(X) \leq p\binom{e+1}{p+1} - \binom{e}{p-1} \quad \textit{for all } p \leq e \,,$$

and the following are equivalent:

(a) X is a del Pezzo variety;

(c) one $\beta_{p,1}(X)$ achieves the maximum for some $1 \leq p \leq e-1$;

(d) all the $\beta_{p,1}(X)$ *for* $1 \leq p \leq e-1$ achieve the maximum;

We also characterize Fano varieties as follows:

Theorem [Han-K, 2015]

Unless X is a variety of minimal degree, then we have

$$\beta_{p,1}(X) \leq p\binom{e+1}{p+1} - \binom{e}{p-1} \quad \textit{for all } p \leq e \, ,$$

and the following are equivalent:

(a) X is a del Pezzo variety;

(c) one $\beta_{p,1}(X)$ achieves the maximum for some $1 \leq p \leq e-1$;

(d) all the $\beta_{p,1}(X)$ *for* $1 \leq p \leq e-1$ achieve the maximum;

- Elimination mapping cone sequence and the partial elimination ideal theory due to M. Green are very useful to prove the above fundamental inequality on the Betti numbers $\beta_{i,1}(X)$ and $\beta_{i,1}(X_q)$.

Elimination mapping cone sequence

Let $S = k[x_1, \dots, x_{n+e}] \subset R = k[x_0, x_1 \dots, x_{n+e}]$

Let M be a graded R-module (so, M is also a graded S-module).

Then, we have a natural long exact sequence:

$$\mathrm{Tor}_i^R(M)_{i+j} \to \mathrm{Tor}_{i-1}^S(M)_{i-1+j} \stackrel{\times x_0}{\to} \mathrm{Tor}_{i-1}^S(M)_{i-1+j+1} \to \mathrm{Tor}_{i-1}^R(M)_{i-1+j+1}$$

whose connecting homomorphism is induced by the multiplication map $\times x_0 : M(-1) \to M$.

- This long exact sequence is useful to study the syzygies of projections.
- We can prove that if X satisfies property N_p, then there is no $p+2$-secant p-plane to X. (A proof is on the blackboard!)

- Elimination mapping cone sequence and the partial elimination ideal theory due to M. Green are very useful to prove the above fundamental inequality on the Betti numbers $\beta_{i,1}(X)$ and $\beta_{i,1}(X_q)$.

▶ **Elimination mapping cone sequence**

Let $S = k[x_1, \ldots, x_{n+e}] \subset R = k[x_0, x_1 \ldots, x_{n+e}]$

Let M be a graded R-module (so, M is also a graded S-module).

Then, we have a natural long exact sequence:

$$\mathrm{Tor}_i^R(M)_{i+j} \to \mathrm{Tor}_{i-1}^S(M)_{i-1+j} \overset{\times x_0}{\to} \mathrm{Tor}_{i-1}^S(M)_{i-1+j+1} \to \mathrm{Tor}_{i-1}^R(M)_{i-1+j+1}$$

whose connecting homomorphism is induced by the multiplication map $\times x_0 : M(-1) \to M$.

- This long exact sequence is useful to study the syzygies of projections.
- We can prove that if X satisfies property N_p, then there is no $p+2$-secant p-plane to X. (A proof is on the blackboard!)

- Elimination mapping cone sequence and the partial elimination ideal theory due to M. Green are very useful to prove the above fundamental inequality on the Betti numbers $\beta_{i,1}(X)$ and $\beta_{i,1}(X_q)$.

▶ Elimination mapping cone sequence

Let $S = k[x_1, \ldots, x_{n+e}] \subset R = k[x_0, x_1 \ldots, x_{n+e}]$
Let M be a graded R-module (so, M is also a graded S-module).
Then, we have a natural long exact sequence:

$$\mathrm{Tor}_i^R(M)_{i+j} \to \mathrm{Tor}_{i-1}^S(M)_{i-1+j} \xrightarrow{\times x_0} \mathrm{Tor}_{i-1}^S(M)_{i-1+j+1} \to \mathrm{Tor}_{i-1}^R(M)_{i-1+j+1}$$

whose connecting homomorphism is induced by the multiplication map $\times x_0 : M(-1) \to M$.

- This long exact sequence is useful to study the syzygies of projections.
- We can prove that if X satisfies property N_p, then there is no $p+2$-secant p-plane to X. (A proof is on the blackboard!)

- Elimination mapping cone sequence and the partial elimination ideal theory due to M. Green are very useful to prove the above fundamental inequality on the Betti numbers $\beta_{i,1}(X)$ and $\beta_{i,1}(X_q)$.

▶ **Elimination mapping cone sequence**

Let $S = k[x_1, \ldots, x_{n+e}] \subset R = k[x_0, x_1 \ldots, x_{n+e}]$
Let M be a graded R-module (so, M is also a graded S-module).
Then, we have a natural long exact sequence:

$$\mathrm{Tor}_i^R(M)_{i+j} \to \mathrm{Tor}_{i-1}^S(M)_{i-1+j} \stackrel{\times x_0}{\to} \mathrm{Tor}_{i-1}^S(M)_{i-1+j+1} \to \mathrm{Tor}_{i-1}^R(M)_{i-1+j+1}$$

whose connecting homomorphism is induced by the multiplication map $\times x_0 : M(-1) \to M$.

- This long exact sequence is useful to study the syzygies of projections.
- We can prove that if X satisfies property N_p, then there is no $p+2$-secant p-plane to X. (A proof is on the blackboard!)

- Elimination mapping cone sequence and the partial elimination ideal theory due to M. Green are very useful to prove the above fundamental inequality on the Betti numbers $\beta_{i,1}(X)$ and $\beta_{i,1}(X_q)$.

▶ Elimination mapping cone sequence

Let $S = k[x_1, \ldots, x_{n+e}] \subset R = k[x_0, x_1 \ldots, x_{n+e}]$
Let M be a graded R-module (so, M is also a graded S-module).
Then, we have a natural long exact sequence:

$$\mathrm{Tor}_i^R(M)_{i+j} \to \mathrm{Tor}_{i-1}^S(M)_{i-1+j} \stackrel{\times x_0}{\to} \mathrm{Tor}_{i-1}^S(M)_{i-1+j+1} \to \mathrm{Tor}_{i-1}^R(M)_{i-1+j+1}$$

whose connecting homomorphism is induced by the multiplication map $\times x_0 : M(-1) \to M$.

- This long exact sequence is useful to study the syzygies of projections.
- We can prove that if X satisfies property N_p, then there is no $p+2$-secant p-plane to X. (A proof is on the blackboard!)

- Elimination mapping cone sequence and the partial elimination ideal theory due to M. Green are very useful to prove the above fundamental inequality on the Betti numbers $\beta_{i,1}(X)$ and $\beta_{i,1}(X_q)$.

▶ **Elimination mapping cone sequence**

Let $S = k[x_1, \ldots, x_{n+e}] \subset R = k[x_0, x_1 \ldots, x_{n+e}]$
Let M be a graded R-module (so, M is also a graded S-module).
Then, we have a natural long exact sequence:
$\mathrm{Tor}_i^R(M)_{i+j} \to \mathrm{Tor}_{i-1}^S(M)_{i-1+j} \overset{\times x_0}{\to} \mathrm{Tor}_{i-1}^S(M)_{i-1+j+1} \to \mathrm{Tor}_{i-1}^R(M)_{i-1+j+1}$
whose connecting homomorphism is induced by the multiplication map $\times x_0 : M(-1) \to M$.

- This long exact sequence is useful to study the syzygies of projections.
- We can prove that if X satisfies property N_p, then there is no $p+2$-secant p-plane to X. (A proof is on the blackboard!)

Varieties of minimal degree

$X^n \subset \mathbb{P}^{n+e}$: a variety (not necessarily smooth) of degree d.
Note that $d \geq e+1$. X is called "minimal degree variety" if $d = e+1$.

▶ The simplest Betti table with $\beta_{i,1} = i \cdot \binom{e+1}{i+1}$:

	0	1	2	3	$\cdots$	$i-1$	i	$i+1$	$\cdots$	e
0	1	$-$	$-$	$-$	$\cdots$	$-$	$-$	$-$	$\cdots$	$-$
1	$-$	$\beta_{1,1}$	$\beta_{2,1}$	$\beta_{3,1}$	$\cdots$	$\beta_{i-1,1}$	$\beta_{i,1}$	$\beta_{i+1,1}$	$\cdots$	$\beta_{e,1}$

Table 1 minimal degree varieties

▶ A VMD has a rational normal curve section and they have the same Betti table.

Varieties of minimal degree

$X^n \subset \mathbb{P}^{n+e}$: a variety (not necessarily smooth) of degree d.
Note that $d \geq e+1$. X is called "minimal degree variety" if $d = e+1$.

- The simplest Betti table with $\beta_{i,1} = i \cdot \binom{e+1}{i+1}$:

	0	1	2	3	$\cdots$	$i-1$	i	$i+1$	$\cdots$	e
0	1	$-$	$-$	$-$	$\cdots$	$-$	$-$	$-$	$\cdots$	$-$
1	$-$	$\beta_{1,1}$	$\beta_{2,1}$	$\beta_{3,1}$	$\cdots$	$\beta_{i-1,1}$	$\beta_{i,1}$	$\beta_{i+1,1}$	$\cdots$	$\beta_{e,1}$

Table 1 minimal degree varieties

- A VMD has a rational normal curve section and they have the same Betti table.

$X^n \subset \mathbb{P}^{n+e}$: a variety (not necessarily smooth) of degree d.
Note that $d \geq e+1$. X is called "minimal degree variety" if $d = e+1$.

- The simplest Betti table with $\beta_{i,1} = i \cdot \binom{e+1}{i+1}$:

	0	1	2	3	$\cdots$	$i-1$	i	$i+1$	$\cdots$	e
0	1	–	–	–	$\cdots$	–	–	–	$\cdots$	–
1	–	$\beta_{1,1}$	$\beta_{2,1}$	$\beta_{3,1}$	$\cdots$	$\beta_{i-1,1}$	$\beta_{i,1}$	$\beta_{i+1,1}$	$\cdots$	$\beta_{e,1}$

Table 1 minimal degree varieties

- A VMD has a rational normal curve section and they have the same Betti table.

On the other hand, P. del Pezzo(1886) and E. Bertini(1907) classified all varieties of minimal degree:

- X is of minimal degree iff X is (a cone of) one of the following;

(a) a quadric hypersurface;

(b) a Veronese surface $\nu_2(\mathbb{P}^2)$ in $\mathbb{P}^5$;

(c) a rational normal scroll, i.e. $\mathbb{P}(\mathcal{E}) \hookrightarrow \mathbb{P}^{\Sigma a_i + d}$, where $\mathcal{E} \simeq \bigoplus_{i=0}^{d} \mathcal{O}_{\mathbb{P}^1}(a_i)$, $a_i \geq 1$.

Also See the paper "On Varieties of Minimal Degree (A centennial Account)-1987" due to D. Eisenbud and J. Harris.

On the other hand, P. del Pezzo(1886) and E. Bertini(1907) classified all varieties of minimal degree:

- X is of minimal degree iff X is (a cone of) one of the following;

(a) a quadric hypersurface;

(b) a Veronese surface $\nu_2(\mathbb{P}^2)$ in $\mathbb{P}^5$;

(c) a rational normal scroll, i.e. $\mathbb{P}(\mathcal{E}) \hookrightarrow \mathbb{P}^{\Sigma a_i + d}$, where $\mathcal{E} \simeq \bigoplus_{i=0}^{d} \mathcal{O}_{\mathbb{P}^1}(a_i)$, $a_i \geq 1$.

Also See the paper "On Varieties of Minimal Degree (A centennial Account)-1987" due to D. Eisenbud and J. Harris.

On the other hand, P. del Pezzo(1886) and E. Bertini(1907) classified all varieties of minimal degree:

- X is of minimal degree iff X is (a cone of) one of the following;

(a) a quadric hypersurface;

(b) a Veronese surface $\nu_2(\mathbb{P}^2)$ in $\mathbb{P}^5$;

(c) a rational normal scroll, i.e. $\mathbb{P}(\mathcal{E}) \hookrightarrow \mathbb{P}^{\Sigma a_i + d}$, where $\mathcal{E} \simeq \bigoplus_{i=0}^{d} \mathcal{O}_{\mathbb{P}^1}(a_i), a_i \geq 1$.

Also See the paper "On Varieties of Minimal Degree (A centennial Account)-1987" due to D. Eisenbud and J. Harris.

del Pezzo varieties

X is called a del Pezzo variety if $d = e + 2$ and $\text{depth}(X) = n + 1$.

▶ The (next-to-simplest) Betti table of a del Pezzo variety:

	0	1	2	3	$\cdots$	i	$\cdots$	$e-1$	e
0	1	–	–	–	$\cdots$	–	$\cdots$	–	–
1	–	$\beta_{1,1}$	$\beta_{2,1}$	$\beta_{3,1}$	$\cdots$	$\beta_{i,1}$	$\cdots$	$\beta_{e-1,1}$	–
2	–	–	–	–	$\cdots$	–	–	–	$\beta_{e,2} = 1$

Table 2 del Pezzo variety with $\beta_{i,1}(X) = i\binom{e+1}{i+1} - \binom{e}{i-1}$.

▶ A del Pezzo variety has an ellitic normal curve section or a rational nodal curve section and they have the same Betti table.

del Pezzo varieties

X is called a del Pezzo variety if $d = e + 2$ and $\mathrm{depth}(X) = n + 1$.

- The (next-to-simplest) Betti table of a del Pezzo variety:

	0	1	2	3	$\cdots$	i	$\cdots$	$e-1$	e
0	1	$-$	$-$	$-$	$\cdots$	$-$	$\cdots$	$-$	$-$
1	$-$	$\beta_{1,1}$	$\beta_{2,1}$	$\beta_{3,1}$	$\cdots$	$\beta_{i,1}$	$\cdots$	$\beta_{e-1,1}$	$-$
2	$-$	$-$	$-$	$-$	$\cdots$	$-$	$-$	$-$	$\beta_{e,2} = 1$

Table 2 del Pezzo variety with $\beta_{i,1}(X) = i\binom{e+1}{i+1} - \binom{e}{i-1}$.

- A del Pezzo variety has an ellitic normal curve section or a rational nodal curve section and they have the same Betti table.

del Pezzo varieties

X is called a del Pezzo variety if $d = e + 2$ and $\mathrm{depth}(X) = n + 1$.

- The (next-to-simplest) Betti table of a del Pezzo variety:

	0	1	2	3	$\cdots$	i	$\cdots$	$e-1$	e
0	1	–	–	–	$\cdots$	–	$\cdots$	–	–
1	–	$\beta_{1,1}$	$\beta_{2,1}$	$\beta_{3,1}$	$\cdots$	$\beta_{i,1}$	$\cdots$	$\beta_{e-1,1}$	–
2	–	–	–	–	$\cdots$	–	–	–	$\beta_{e,2} = 1$

Table 2 del Pezzo variety with $\beta_{i,1}(X) = i\binom{e+1}{i+1} - \binom{e}{i-1}$.

- A del Pezzo variety has an ellitic normal curve section or a rational nodal curve section and they have the same Betti table.

del Pezzo varieties

- T. Fujita classified smooth del Pezzo varieties into 8 types.
- T. Fujita also showed that (singular) normal del Pezzo cases are divisors of some specific type in rational normal scrolls.
- (M. Brodmann-P. Schenzel) Every non-normal del Pezzo variety X comes from outer projection of a minimal degree variety $\widetilde{X}$ from a point q in $Sec(\widetilde{X}) \setminus \widetilde{X}$ satisfying $\dim \Sigma_q(\widetilde{X}) = \dim \widetilde{X} - 1$ where the secant locus

$$\Sigma_q(\widetilde{X}) := \{x \in \widetilde{X} \mid \pi_q^{-1}(\pi_q(x)) \textit{ has length at least } 2\}.$$

- Thus, a non-normal del Pezzo variety has a rational nodal curve section.

del Pezzo varieties

- T. Fujita classified smooth del Pezzo varieties into 8 types.
- T. Fujita also showed that (singular) normal del Pezzo cases are divisors of some specific type in rational normal scrolls.
- (M. Brodmann-P. Schenzel) Every non-normal del Pezzo variety X comes from outer projection of a minimal degree variety $\widetilde{X}$ from a point q in $\mathrm{Sec}(\widetilde{X}) \setminus \widetilde{X}$ satisfying $\dim \Sigma_q(\widetilde{X}) = \dim \widetilde{X} - 1$ where the secant locus

$$\Sigma_q(\widetilde{X}) := \{x \in \widetilde{X} \mid \pi_q^{-1}(\pi_q(x)) \textit{ has length at least } 2\}.$$

- Thus, a non-normal del Pezzo variety has a rational nodal curve section.

del Pezzo varieties

- T. Fujita classified smooth del Pezzo varieties into 8 types.
- T. Fujita also showed that (singular) normal del Pezzo cases are divisors of some specific type in rational normal scrolls.
- (M. Brodmann-P. Schenzel) Every non-normal del Pezzo variety X comes from outer projection of a minimal degree variety $\widetilde{X}$ from a point q in $\mathrm{Sec}(\widetilde{X}) \setminus \widetilde{X}$ satisfying $\dim \Sigma_q(\widetilde{X}) = \dim \widetilde{X} - 1$ where the secant locus

$$\Sigma_q(\widetilde{X}) := \{x \in \widetilde{X} \mid \pi_q^{-1}(\pi_q(x)) \textit{ has length at least } 2\}.$$

- Thus, a non-normal del Pezzo variety has a rational nodal curve section.

del Pezzo varieties

- T. Fujita classified smooth del Pezzo varieties into 8 types.
- T. Fujita also showed that (singular) normal del Pezzo cases are divisors of some specific type in rational normal scrolls.
- (M. Brodmann-P. Schenzel) Every non-normal del Pezzo variety X comes from outer projection of a minimal degree variety $\widetilde{X}$ from a point q in $\mathrm{Sec}(\widetilde{X}) \setminus \widetilde{X}$ satisfying $\dim \Sigma_q(\widetilde{X}) = \dim \widetilde{X} - 1$ where the secant locus

$$\Sigma_q(\widetilde{X}) := \{x \in \widetilde{X} \mid \pi_q^{-1}(\pi_q(x)) \textit{ has length at least } 2\ \}.$$

- Thus, a non-normal del Pezzo variety has a rational nodal curve section.

Classification of non-normal del Pezzo varieties

- M. Brodmann and E. Park showed that $\exists$ only 8 types of *non-normal* del Pezzo which are not cones:
  (i) Outer projections of a rational normal curve $C \subset \mathbb{P}^a$ $(a > 2)$ from $q \in \mathrm{Sec}(C)$,
  (ii) Outer projections of the Veronese surface $\upsilon_2(\mathbb{P}^2) \subset \mathbb{P}^5$ from $q \in \mathrm{Sec}(\upsilon_2(\mathbb{P}^2))$,
  (iii) Outer projections of a smooth cubic surface scroll $S(1,2)$ in $\mathbb{P}^4$,
  (iv) Outer projections of a smooth rational normal scroll $S(1,b)$ in $\mathbb{P}^{b+2}$ $(b > 2)$ from $q \in \mathit{Join}(S(1), S(1,b))$,
  (v) Outer projections of a smooth quartic surface scroll $S(2,2)$ in $\mathbb{P}^5$ from $q \in \mathbb{P}^2 \times \mathbb{P}^1$,
  (vi) Outer projections of a smooth surface scroll $S(2,b)$ in $\mathbb{P}^{b+3}$ $(b > 2)$ from $q \in \langle S(2) \rangle$,
  (vii) Outer projections of a smooth 3-fold scroll $S(1,1,1)$ in $\mathbb{P}^5$,
  (viii) Outer projections of a smooth 3-fold scroll $S(1,1,c)$ in $\mathbb{P}^{c+4}$ with $c > 1$ from $q \in \langle S(1,1) \rangle$.

Classification of non-normal del Pezzo varieties

- M. Brodmann and E. Park showed that $\exists$ only 8 types of *non-normal* del Pezzo which are not cones:
  (i) Outer projections of a rational normal curve $C \subset \mathbb{P}^a$ $(a > 2)$ from $q \in \mathrm{Sec}(C)$,
  (ii) Outer projections of the Veronese surface $\upsilon_2(\mathbb{P}^2) \subset \mathbb{P}^5$ from $q \in \mathrm{Sec}(\upsilon_2(\mathbb{P}^2))$,
  (iii) Outer projections of a smooth cubic surface scroll $S(1,2)$ in $\mathbb{P}^4$,
  (iv) Outer projections of a smooth rational normal scroll $S(1,b)$ in $\mathbb{P}^{b+2}$ $(b > 2)$ from $q \in Join(S(1), S(1,b))$,
  (v) Outer projections of a smooth quartic surface scroll $S(2,2)$ in $\mathbb{P}^5$ from $q \in \mathbb{P}^2 \times \mathbb{P}^1$,
  (vi) Outer projections of a smooth surface scroll $S(2,b)$ in $\mathbb{P}^{b+3}$ $(b > 2)$ from $q \in \langle S(2) \rangle$,
  (vii) Outer projections of a smooth 3-fold scroll $S(1,1,1)$ in $\mathbb{P}^5$,
  (viii) Outer projections of a smooth 3-fold scroll $S(1,1,c)$ in $\mathbb{P}^{c+4}$ with $c > 1$ from $q \in \langle S(1,1) \rangle$.

Classification of non-normal del Pezzo varieties

- M. Brodmann and E. Park showed that $\exists$ only 8 types of *non-normal* del Pezzo which are not cones:
  (i) Outer projections of a rational normal curve $C \subset \mathbb{P}^a$ $(a > 2)$ from $q \in \mathrm{Sec}(C)$,
  (ii) Outer projections of the Veronese surface $\upsilon_2(\mathbb{P}^2) \subset \mathbb{P}^5$ from $q \in \mathrm{Sec}(\upsilon_2(\mathbb{P}^2))$,
  (iii) Outer projections of a smooth cubic surface scroll $S(1,2)$ in $\mathbb{P}^4$,
  (iv) Outer projections of a smooth rational normal scroll $S(1,b)$ in $\mathbb{P}^{b+2}$ $(b > 2)$ from $q \in Join(S(1), S(1,b))$,
  (v) Outer projections of a smooth quartic surface scroll $S(2,2)$ in $\mathbb{P}^5$ from $q \in \mathbb{P}^2 \times \mathbb{P}^1$,
  (vi) Outer projections of a smooth surface scroll $S(2,b)$ in $\mathbb{P}^{b+3}$ $(b > 2)$ from $q \in \langle S(2) \rangle$,
  (vii) Outer projections of a smooth 3-fold scroll $S(1,1,1)$ in $\mathbb{P}^5$,
  (viii) Outer projections of a smooth 3-fold scroll $S(1,1,c)$ in $\mathbb{P}^{c+4}$ with $c > 1$ from $q \in \langle S(1,1) \rangle$.

Classification of non-normal del Pezzo varieties

- M. Brodmann and E. Park showed that $\exists$ only 8 types of *non-normal* del Pezzo which are not cones:
  (i) Outer projections of a rational normal curve $C \subset \mathbb{P}^a$ $(a > 2)$ from $q \in \mathrm{Sec}(C)$,
  (ii) Outer projections of the Veronese surface $\upsilon_2(\mathbb{P}^2) \subset \mathbb{P}^5$ from $q \in \mathrm{Sec}(\upsilon_2(\mathbb{P}^2))$,
  (iii) Outer projections of a smooth cubic surface scroll $S(1,2)$ in $\mathbb{P}^4$,
  (iv) Outer projections of a smooth rational normal scroll $S(1,b)$ in $\mathbb{P}^{b+2}$ $(b > 2)$ from $q \in Join(S(1), S(1,b))$,
  (v) Outer projections of a smooth quartic surface scroll $S(2,2)$ in $\mathbb{P}^5$ from $q \in \mathbb{P}^2 \times \mathbb{P}^1$,
  (vi) Outer projections of a smooth surface scroll $S(2,b)$ in $\mathbb{P}^{b+3}$ $(b > 2)$ from $q \in \langle S(2) \rangle$,
  (vii) Outer projections of a smooth 3-fold scroll $S(1,1,1)$ in $\mathbb{P}^5$,
  (viii) Outer projections of a smooth 3-fold scroll $S(1,1,c)$ in $\mathbb{P}^{c+4}$ with $c > 1$ from $q \in \langle S(1,1) \rangle$.

Asymptotic behavior of $\beta_{p,1}$ for smooth curves

Let C be a smooth curve of genus g and the gonality g_0.
Suppose $\deg(\mathcal{L}) \geq 4g-3$ and $r = h^0(C,\mathcal{L})-1$.

(a) $\beta_{p,1}(C) \neq 0 \Longleftrightarrow 1 \leq p \leq r-g_0$ (Ein-Lazarsfeld);
(b) $\beta_{p,2}(C) \neq 0 \Longleftrightarrow r-g \leq p \leq r-1$ (Green and Schreyer).

	0	1	$\cdots$	$r-g-1$	$r-g$	$\cdots$	$r-g_0$	$\cdots$	$r-1$
0	1	–	$\cdots$	–	$\cdots$	–	–	–	–
1	–	$\beta_{1,1}$	$\cdots$	$\beta_{r-g-1,1}$	$\beta_{r-g,1}$	$\cdots$	$\beta_{r-g_0,1}$	–	–
2	–	–	$\cdots$	–	$\beta_{r-g,2}$	$\cdots$	$\cdots$		$\beta_{r-1,2}$

► We can also get the upper bound of $\beta_{p,1}$ by using inner projection method as follows:

$$\beta_{p,1}(C) \leq p\binom{e+1}{p+1} + \binom{e+1}{p+1} - \binom{e+2-g_0}{p+1} - (g_0-1)\binom{e+1}{p}.$$

Asymptotic behavior of $\beta_{p,1}$ for smooth curves

Let C be a smooth curve of genus g and the gonality g_0.
Suppose $\deg(\mathcal{L}) \geq 4g-3$ and $r = h^0(C,\mathcal{L})-1$.

(a) $\beta_{p,1}(C) \neq 0 \Longleftrightarrow 1 \leq p \leq r-g_0$ (Ein-Lazarsfeld);

(b) $\beta_{p,2}(C) \neq 0 \Longleftrightarrow r-g \leq p \leq r-1$ (Green and Schreyer).

	0	1	$\cdots$	$r-g-1$	$r-g$	$\cdots$	$r-g_0$	$\cdots$	$r-1$
0	1	$-$	$\cdots$	$-$	$\cdots$	$-$	$-$	$-$	$-$
1	$-$	$\beta_{1,1}$	$\cdots$	$\beta_{r-g-1,1}$	$\beta_{r-g,1}$	$\cdots$	$\beta_{r-g_0,1}$	$-$	$-$
2	$-$	$-$	$\cdots$	$-$	$\beta_{r-g,2}$	$\cdots$	$\cdots$		$\beta_{r-1,2}$

- We can also get the upper bound of $\beta_{p,1}$ by using inner projection method as follows:

$$\beta_{p,1}(C) \leq p\binom{e+1}{p+1} + \binom{e+1}{p+1} - \binom{e+2-g_0}{p+1} - (g_0-1)\binom{e+1}{p}.$$

Asymptotic behavior of $\beta_{p,1}$ for smooth curves

Let C be a smooth curve of genus g and the gonality g_0.
Suppose $\deg(\mathcal{L}) \geq 4g-3$ and $r = h^0(C,\mathcal{L}) - 1$.

(a) $\beta_{p,1}(C) \neq 0 \iff 1 \leq p \leq r - g_0$ (Ein-Lazarsfeld);
(b) $\beta_{p,2}(C) \neq 0 \iff r - g \leq p \leq r - 1$ (Green and Schreyer).

	0	1	$\cdots$	$r-g-1$	$r-g$	$\cdots$	$r-g_0$	$\cdots$	$r-1$
0	1	–	$\cdots$	–	$\cdots$	–	–	–	–
1	–	$\beta_{1,1}$	$\cdots$	$\beta_{r-g-1,1}$	$\beta_{r-g,1}$	$\cdots$	$\beta_{r-g_0,1}$	–	–
2	–	–	$\cdots$	–	$\beta_{r-g,2}$	$\cdots$	$\cdots$		$\beta_{r-1,2}$

- We can also get the upper bound of $\beta_{p,1}$ by using inner projection method as follows:

$$\beta_{p,1}(C) \leq p\binom{e+1}{p+1} + \binom{e+1}{p+1} - \binom{e+2-g_0}{p+1} - (g_0-1)\binom{e+1}{p}.$$

Asymptotic behavior of $\beta_{p,1}$ for smooth curves

Let C be a smooth curve of genus g and the gonality g_0.
Suppose $\deg(\mathcal{L}) \geq 4g-3$ and $r = h^0(C,\mathcal{L}) - 1$.

(a) $\beta_{p,1}(C) \neq 0 \Longleftrightarrow 1 \leq p \leq r - g_0$ (Ein-Lazarsfeld);

(b) $\beta_{p,2}(C) \neq 0 \Longleftrightarrow r - g \leq p \leq r - 1$ (Green and Schreyer).

	0	1	$\cdots$	$r-g-1$	$r-g$	$\cdots$	$r-g_0$	$\cdots$	$r-1$
0	1	–	$\cdots$	–	$\cdots$	–	–	–	–
1	–	$\beta_{1,1}$	$\cdots$	$\beta_{r-g-1,1}$	$\beta_{r-g,1}$	$\cdots$	$\beta_{r-g_0,1}$	–	–
2	–	–	$\cdots$	–	$\beta_{r-g,2}$	$\cdots$	$\cdots$		$\beta_{r-1,2}$

► We can also get the upper bound of $\beta_{p,1}$ by using inner projection method as follows:

$$\beta_{p,1}(C) \leq p\binom{e+1}{p+1} + \binom{e+1}{p+1} - \binom{e+2-g_0}{p+1} - (g_0-1)\binom{e+1}{p}.$$

Asymptotic behavior of $\beta_{p,1}$ for smooth curves

Let C be a smooth curve of genus g and the gonality g_0.
Suppose $\deg(\mathcal{L}) \geq 4g-3$ and $r = h^0(C,\mathcal{L})-1$.

(a) $\beta_{p,1}(C) \neq 0 \iff 1 \leq p \leq r-g_0$ (Ein-Lazarsfeld);
(b) $\beta_{p,2}(C) \neq 0 \iff r-g \leq p \leq r-1$ (Green and Schreyer).

	0	1	$\cdots$	$r-g-1$	$r-g$	$\cdots$	$r-g_0$	$\cdots$	$r-1$
0	1	–	$\cdots$	–	$\cdots$	–	–	–	–
1	–	$\beta_{1,1}$	$\cdots$	$\beta_{r-g-1,1}$	$\beta_{r-g,1}$	$\cdots$	$\beta_{r-g_0,1}$	–	–
2	–	–	$\cdots$	–	$\beta_{r-g,2}$	$\cdots$	$\cdots$		$\beta_{r-1,2}$

► We can also get the upper bound of $\beta_{p,1}$ by using inner projection method as follows:

$$\beta_{p,1}(C) \leq p\binom{e+1}{p+1} + \binom{e+1}{p+1} - \binom{e+2-g_0}{p+1} - (g_0-1)\binom{e+1}{p}.$$

The graded Betti table of R/I_X

	0	1	2	3	$\cdots$	$i-1$	i	$i+1$	$\cdots$	$\triangle$
0	1	$-$	$-$	$-$	$\cdots$	$-$	$-$	$-$	$\cdots$	$-$
1	$-$	$\beta_{1,1}$	$\beta_{2,1}$	$\beta_{3,1}$	$\cdots$	$\beta_{i-1,1}$	$\beta_{i,1}$	$\beta_{i+1,1}$	$\cdots$	$\beta_{\triangle,1}$
2	$-$	$\beta_{1,2}$	$\beta_{2,2}$	$\beta_{3,2}$	$\cdots$	$\beta_{i-1,2}$	$\beta_{i,2}$	$\beta_{i+1,2}$	$\cdots$	$\beta_{\triangle,2}$
$\vdots$	$-$	$-$	$\dots$	$-$	$\ddots$	$\dots$	$\vdots$	$\vdots$	$\dots$	$\ddots$
j	$-$	$\beta_{1,j}$	$\beta_{2,j}$	$\beta_{3,j}$	$\cdots$	$\beta_{i-1,j}$	$\beta_{i,j}$	$\beta_{i+1,j}$	$\cdots$	$\beta_{\triangle,j}$
$\vdots$			$\dots$	$-$	$\ddots$	$\dots$	$\vdots$	$\vdots$	$\dots$	$\ddots$
$\square$	$-$	$\beta_{1,\square}$	$\beta_{2,\square}$	$\beta_{3,\square}$	$\cdots$	$\beta_{i-1,\square}$	$\beta_{i,\square}$	$\beta_{i+1,\square}$	$\cdots$	$\beta_{\triangle,\square}$

In the first linear strand, we summarize the following facts:

- $\beta_{p,1} = 0$ for $p > e$;
- The maximal upper bounds for a VMD;
- The next to maximal upper bounds for a del Pezzo.

The graded Betti table of R/I_X

	0	1	2	3	$\cdots$	$i-1$	i	$i+1$	$\cdots$	$\triangle$
0	1	–	–	–	$\cdots$	–	–	–	$\cdots$	–
1	–	$\beta_{1,1}$	$\beta_{2,1}$	$\beta_{3,1}$	$\cdots$	$\beta_{i-1,1}$	$\beta_{i,1}$	$\beta_{i+1,1}$	$\cdots$	$\beta_{\triangle,1}$
2	–	$\beta_{1,2}$	$\beta_{2,2}$	$\beta_{3,2}$	$\cdots$	$\beta_{i-1,2}$	$\beta_{i,2}$	$\beta_{i+1,2}$	$\cdots$	$\beta_{\triangle,2}$
$\vdots$	–	–	$\dots$	–	$\ddots$	$\dots$	$\vdots$	$\vdots$	$\dots$	$\ddots$
j	–	$\beta_{1,j}$	$\beta_{2,j}$	$\beta_{3,j}$	$\cdots$	$\beta_{i-1,j}$	$\beta_{i,j}$	$\beta_{i+1,j}$	$\cdots$	$\beta_{\triangle,j}$
$\vdots$			$\dots$	–	$\ddots$	$\dots$	$\vdots$	$\vdots$	$\dots$	$\ddots$
$\square$	–	$\beta_{1,\square}$	$\beta_{2,\square}$	$\beta_{3,\square}$	$\cdots$	$\beta_{i-1,\square}$	$\beta_{i,\square}$	$\beta_{i+1,\square}$	$\cdots$	$\beta_{\triangle,\square}$

In the first linear strand, we summarize the following facts:

- $\beta_{p,1} = 0$ for $p > e$;
- The maximal upper bounds for a VMD;
- The next to maximal upper bounds for a del Pezzo.

The graded Betti table of R/I_X

	0	1	2	3	$\cdots$	$i-1$	i	$i+1$	$\cdots$	$\triangle$
0	1	–	–	–	$\cdots$	–	–	–	$\cdots$	–
1	–	$\beta_{1,1}$	$\beta_{2,1}$	$\beta_{3,1}$	$\cdots$	$\beta_{i-1,1}$	$\beta_{i,1}$	$\beta_{i+1,1}$	$\cdots$	$\beta_{\triangle,1}$
2	–	$\beta_{1,2}$	$\beta_{2,2}$	$\beta_{3,2}$	$\cdots$	$\beta_{i-1,2}$	$\beta_{i,2}$	$\beta_{i+1,2}$	$\cdots$	$\beta_{\triangle,2}$
$\vdots$	–	–	$\ldots$	–	$\ddots$	$\ldots$	$\vdots$	$\vdots$	$\ldots$	$\ddots$
j	–	$\beta_{1,j}$	$\beta_{2,j}$	$\beta_{3,j}$	$\cdots$	$\beta_{i-1,j}$	$\beta_{i,j}$	$\beta_{i+1,j}$	$\cdots$	$\beta_{\triangle,j}$
$\vdots$			$\ldots$	–	$\ddots$	$\ldots$	$\vdots$	$\vdots$	$\ldots$	$\ddots$
$\square$	–	$\beta_{1,\square}$	$\beta_{2,\square}$	$\beta_{3,\square}$	$\cdots$	$\beta_{i-1,\square}$	$\beta_{i,\square}$	$\beta_{i+1,\square}$	$\cdots$	$\beta_{\triangle,\square}$

In the first linear strand, we summarize the following facts:

- $\beta_{p,1} = 0$ for $p > e$;
- The maximal upper bounds for a VMD;
- The next to maximal upper bounds for a del Pezzo.

The structure of the Betti table II (Regularity)

- Let $X^n \subset \mathbb{P}^{n+e}$ be a non-degenerate projective variety of dim n, codim e, and degree d, and H be a general hyperplane section.

Definition

- X is called m-regular if the following two conditions hold:
  1. $H^0(\mathcal{O}_{\mathbb{P}^{n+e}}(m-1)) \twoheadrightarrow H^0(\mathcal{O}_X(m-1))$ is surjective, i.e. X is $(m-1)$-normal;
  2. $H^i(\mathcal{O}_X(m-1-i)) = 0$ for all $i \geq 1$, i.e. $\mathcal{O}_X$ is $(m-1)$-regular with respect to $\mathcal{O}_X(1)$.
- $\mathrm{reg}(X)$ (Castelnuovo-Mumford regularity of X) is the smallest number m such that X is m-regular;

Geometric Regularity Bound

$$\mathrm{reg}(X) \leq d - e + 1 \text{ (Eisenbud-Goto conjecture).}$$

The structure of the Betti table II (Regularity)

- Let $X^n \subset \mathbb{P}^{n+e}$ be a non-degenerate projective variety of dim n, codim e, and degree d, and H be a general hyperplane section.

Definition

- X is called m-regular if the following two conditions hold:
  1. $H^0(\mathcal{O}_{\mathbb{P}^{n+e}}(m-1)) \twoheadrightarrow H^0(\mathcal{O}_X(m-1))$ is surjective, i.e. X is $(m-1)$-normal;
  2. $H^i(\mathcal{O}_X(m-1-i)) = 0$ for all $i \geq 1$, i.e. $\mathcal{O}_X$ is $(m-1)$-regular with respect to $\mathcal{O}_X(1)$.
- $\mathrm{reg}(X)$ (Castelnuovo-Mumford regularity of X) is the smallest number m such that X is m-regular;

Geometric Regularity Bound

$$\mathrm{reg}(X) \leq d - e + 1 \text{ (Eisenbud-Goto conjecture).}$$

The structure of the Betti table II (Regularity)

- Let $X^n \subset \mathbb{P}^{n+e}$ be a non-degenerate projective variety of dim n, codim e, and degree d, and H be a general hyperplane section.

Definition

- X is called m-regular if the following two conditions hold:
  1. $H^0(\mathcal{O}_{\mathbb{P}^{n+e}}(m-1)) \twoheadrightarrow H^0(\mathcal{O}_X(m-1))$ is surjective, i.e. X is $(m-1)$-normal;
  2. $H^i(\mathcal{O}_X(m-1-i)) = 0$ for all $i \geq 1$, i.e. $\mathcal{O}_X$ is $(m-1)$-regular with respect to $\mathcal{O}_X(1)$.
- $\mathrm{reg}(X)$ (Castelnuovo-Mumford regularity of X) is the smallest number m such that X is m-regular;

Geometric Regularity Bound

$$\mathrm{reg}(X) \leq d - e + 1 \text{ (Eisenbud-Goto conjecture).}$$

The structure of the Betti table II (Regularity)

- Let $X^n \subset \mathbb{P}^{n+e}$ be a non-degenerate projective variety of dim n, codim e, and degree d, and H be a general hyperplane section.

Definition

- X is called m-regular if the following two conditions hold:
  1. $H^0(\mathcal{O}_{\mathbb{P}^{n+e}}(m-1)) \twoheadrightarrow H^0(\mathcal{O}_X(m-1))$ is surjective, i.e. X is $(m-1)$-normal;
  2. $H^i(\mathcal{O}_X(m-1-i)) = 0$ for all $i \geq 1$, i.e. $\mathcal{O}_X$ is $(m-1)$-regular with respect to $\mathcal{O}_X(1)$.
- $\mathrm{reg}(X)$ (Castelnuovo-Mumford regularity of X) is the smallest number m such that X is m-regular;

Geometric Regularity Bound

$$\mathrm{reg}(X) \leq d - e + 1 \text{ (Eisenbud-Goto conjecture).}$$

Proposition (Birational double point formula)
Let $\varphi\colon V^n \to M^{n+1}$ be a morphism of smooth projective varieties such that $\varphi\colon V \to W := \varphi(V) \subset M$ is birational. Then,

$$\varphi^*(K_M + W) - K_V \sim D - E,$$

where D and E are effective divisors on V such that E is φ-exceptional. Moreover, if φ is isomorphic at $x \in V$, then $x \notin \mathrm{Supp}(D - E)$.

Proof. see Lemma 10.2.8. in Positivity in Algebraic Geometry II.

- We apply this formula to a general projection of smooth varieties:

$$\pi_\Lambda : X \twoheadrightarrow X_\Lambda \subset \mathbb{P}^{n+1}, \Lambda = \mathbb{P}^{e-2} \text{ and } \Lambda \cap X = \emptyset.$$

Note that $\deg(X) = \deg(X_\Lambda) = d$.

Proposition (Birational double point formula)
Let $\varphi\colon V^n \to M^{n+1}$ be a morphism of smooth projective varieties such that $\varphi\colon V \to W := \varphi(V) \subset M$ is birational. Then,

$$\varphi^*(K_M + W) - K_V \sim D - E,$$

where D and E are effective divisors on V such that E is φ-exceptional. Moreover, if φ is isomorphic at $x \in V$, then $x \notin \mathrm{Supp}(D - E)$.

Proof. see Lemma 10.2.8. in Positivity in Algebraic Geometry II.

- We apply this formula to a general projection of smooth varieties:

$$\pi_\Lambda : X \twoheadrightarrow X_\Lambda \subset \mathbb{P}^{n+1}, \Lambda = \mathbb{P}^{e-2} \text{ and } \Lambda \cap X = \emptyset.$$

Note that $\deg(X) = \deg(X_\Lambda) = d$.

Proposition (Birational double point formula)
Let $\varphi\colon V^n \to M^{n+1}$ be a morphism of smooth projective varieties such that $\varphi\colon V \to W := \varphi(V) \subset M$ is birational. Then,

$$\varphi^*(K_M + W) - K_V \sim D - E,$$

where D and E are effective divisors on V such that E is φ-exceptional. Moreover, if φ is isomorphic at $x \in V$, then $x \notin \mathrm{Supp}(D - E)$.

Proof. see Lemma 10.2.8. in Positivity in Algebraic Geometry II.

- We apply this formula to a general projection of smooth varieties:

$$\pi_\Lambda : X \twoheadrightarrow X_\Lambda \subset \mathbb{P}^{n+1}, \Lambda = \mathbb{P}^{e-2} \text{ and } \Lambda \cap X = \emptyset.$$

Note that $\deg(X) = \deg(X_\Lambda) = d$.

Proposition (Birational double point formula)
Let $\varphi\colon V^n \to M^{n+1}$ be a morphism of smooth projective varieties such that $\varphi\colon V \to W := \varphi(V) \subset M$ is birational. Then,

$$\varphi^*(K_M + W) - K_V \sim D - E,$$

where D and E are effective divisors on V such that E is φ-exceptional. Moreover, if φ is isomorphic at $x \in V$, then $x \notin \mathrm{Supp}(D - E)$.

Proof. see Lemma 10.2.8. in Positivity in Algebraic Geometry II.

- We apply this formula to a general projection of smooth varieties:

$$\pi_\Lambda : X \twoheadrightarrow X_\Lambda \subset \mathbb{P}^{n+1}, \Lambda = \mathbb{P}^{e-2} \text{ and } \Lambda \cap X = \emptyset.$$

Note that $\deg(X) = \deg(X_\Lambda) = d$.

Theorem [Noma, Park and K-]
Let X be smooth and $\mathcal{L}$ is very ample. We have the following:

$$H^i(X, \mathcal{L}^{\otimes(d-e-i)}) = H^i(\mathcal{O}_X(d-e-i)) = 0, i \geq 1$$

where $d = \deg(X)$ and $e = \mathrm{codim}(X)$ in the embedding of $X \subset \mathbb{P}(H^0(\mathcal{L}))$.

Corollary

$\mathrm{reg} R(X, \mathcal{L}) \leq d - e$, *i.e.* $\beta_{p,q}(R(X)) = 0, \forall q \geq d - e + 1$.

Theorem [Noma, Park and K-]
Let X be smooth and $\mathcal{L}$ is very ample. We have the following:

$$H^i(X, \mathcal{L}^{\otimes(d-e-i)}) = H^i(\mathcal{O}_X(d-e-i)) = 0, i \geq 1$$

where $d = \deg(X)$ and $e = \mathrm{codim}(X)$ in the embedding of $X \subset \mathbb{P}(H^0(\mathcal{L}))$.

Corollary

$\mathrm{reg} R(X, \mathcal{L}) \leq d - e$, *i.e.* $\beta_{p,q}(R(X)) = 0, \forall q \geq d - e + 1$.

Theorem [Noma, Park and K-]
Let X be smooth and $\mathcal{L}$ is very ample. We have the following:

$$H^i(X, \mathcal{L}^{\otimes(d-e-i)}) = H^i(\mathcal{O}_X(d-e-i)) = 0, i \geq 1$$

where $d = \deg(X)$ and $e = \mathrm{codim}(X)$ in the embedding of $X \subset \mathbb{P}(H^0(\mathcal{L}))$.

Corollary

$\mathrm{reg} R(X, \mathcal{L}) \leq d - e$, *i.e.* $\beta_{p,q}(R(X)) = 0, \forall q \geq d - e + 1$.

Duality Theorem for syzygies

Consider the following diagram

$$\begin{array}{ccc} D_\Lambda \subset X & \hookrightarrow & \mathbb{P}^{n+e} \\ \downarrow \pi_\Lambda & & \vdots\, \pi_\Lambda \\ Z_\Lambda \subset X_\Lambda & \hookrightarrow & \mathbb{P}^{n+1}. \end{array}$$

Since π_Λ is finite, so π_Λ-exceptional divisor $E = \varnothing$ and the non-isomorphic locus D_Λ (as a divisor) is linearly equivalent to

$$B_2 := (d-n-2)H - K_X,$$

which is called a double point divisor arising from $\pi_\Lambda : X \twoheadrightarrow X_\Lambda \subset \mathbb{P}^{n+1}$. Let $V \subset H^0(\mathcal{O}_X(B_2))$ be a subspace spanned by geometric sections (i.e. for $s \in V$, $\mathrm{div}(s) = D_\Lambda$ coming from an actual outer projection).

Consider the following diagram

$$\begin{array}{ccc} D_\Lambda \subset X & \hookrightarrow & \mathbb{P}^{n+e} \\ \downarrow \pi_\Lambda & & \vdots\,\pi_\Lambda \\ Z_\Lambda \subset X_\Lambda & \hookrightarrow & \mathbb{P}^{n+1}. \end{array}$$

Since π_Λ is finite, so π_Λ-exceptional divisor $E = \varnothing$ and the non-isomorphic locus D_Λ (as a divisor) is linearly equivalent to

$$B_2 := (d - n - 2)H - K_X,$$

which is called a double point divisor arising from $\pi_\Lambda : X \twoheadrightarrow X_\Lambda \subset \mathbb{P}^{n+1}$. Let $V \subset H^0(\mathcal{O}_X(B_2))$ be a subspace spanned by geometric sections (i.e. for $s \in V$, $\mathrm{div}(s) = D_\Lambda$ coming from an actual outer projection).

Consider the following diagram

$$\begin{array}{ccc} D_\Lambda \subset X & \hookrightarrow & \mathbb{P}^{n+e} \\ \downarrow \pi_\Lambda & & \vdots\ \pi_\Lambda \\ Z_\Lambda \subset X_\Lambda & \hookrightarrow & \mathbb{P}^{n+1}. \end{array}$$

Since π_Λ is finite, so π_Λ-exceptional divisor $E = \varnothing$ and the non-isomorphic locus D_Λ (as a divisor) is linearly equivalent to

$$B_2 := (d - n - 2)H - K_X,$$

which is called a double point divisor arising from $\pi_\Lambda : X \twoheadrightarrow X_\Lambda \subset \mathbb{P}^{n+1}$.
Let $V \subset H^0(\mathcal{O}_X(B_2))$ be a subspace spanned by geometric sections (i.e. for $s \in V$, $\mathrm{div}(s) = D_\Lambda$ coming from an actual outer projection).

Consider the following diagram

$$\begin{array}{ccc} D_\Lambda \subset X & \hookrightarrow & \mathbb{P}^{n+e} \\ \downarrow \pi_\Lambda & & \vdots\, \pi_\Lambda \\ Z_\Lambda \subset X_\Lambda & \hookrightarrow & \mathbb{P}^{n+1}. \end{array}$$

Since π_Λ is finite, so π_Λ-exceptional divisor $E = \varnothing$ and the non-isomorphic locus D_Λ (as a divisor) is linearly equivalent to

$$B_2 := (d - n - 2)H - K_X,$$

which is called a double point divisor arising from $\pi_\Lambda : X \twoheadrightarrow X_\Lambda \subset \mathbb{P}^{n+1}$. Let $V \subset H^0(\mathcal{O}_X(B_2))$ be a subspace spanned by geometric sections (i.e. for $s \in V$, $\mathrm{div}(s) = D_\Lambda$ coming from an actual outer projection).

Consider the following diagram

$$\begin{array}{ccc} D_\Lambda \subset X & \hookrightarrow & \mathbb{P}^{n+e} \\ \downarrow \pi_\Lambda & & \vdots\ \pi_\Lambda \\ Z_\Lambda \subset X_\Lambda & \hookrightarrow & \mathbb{P}^{n+1}. \end{array}$$

Since π_Λ is finite, so π_Λ-exceptional divisor $E = \varnothing$ and the non-isomorphic locus D_Λ (as a divisor) is linearly equivalent to

$$B_2 := (d - n - 2)H - K_X,$$

which is called a double point divisor arising from $\pi_\Lambda : X \twoheadrightarrow X_\Lambda \subset \mathbb{P}^{n+1}$. Let $V \subset H^0(\mathcal{O}_X(B_2))$ be a subspace spanned by geometric sections (i.e. for $s \in V$, $\mathrm{div}(s) = D_\Lambda$ coming from an actual outer projection).

Proposition(Mumford)
V is a basepoint-free subsystem in $H^0(\mathcal{O}_X(B_2))$. In particular,

$$H^i(\mathcal{O}_X(d-i-1)) = 0, \quad i \geq 1.$$

Proof. For $x \in X$, take a general linear space Λ_x such that $\Lambda_x \cap (T_x(X)^n \cup Cone(x,X)^{n+1}) = \emptyset$.Then $\pi_{\Lambda_x} : X \twoheadrightarrow X_{\Lambda_x} \subset \mathbb{P}^{n+1}$ is isomorphic near x and $x \notin D_{\Lambda_x}$ and $B_2 := (d-n-2)H - K_X$ is basepoint-free. Since $(d-1-i)H = K_X + (n+1-i)H + B_2$, by Kodaira vanishing theorem $H^i(\mathcal{O}_X(d-i-1)) = 0, \ i \geq 1.$ □

Remark

$\mathrm{reg}_H(\mathcal{O}_X) \leq d-1$ for a smooth projective variety X.

Proposition(Mumford)
V is a basepoint-free subsystem in $H^0(\mathcal{O}_X(B_2))$. In particular,

$$H^i(\mathcal{O}_X(d-i-1)) = 0, \quad i \geq 1.$$

Proof. For $x \in X$, take a general linear space Λ_x such that $\Lambda_x \cap (T_x(X)^n \cup Cone(x, X)^{n+1}) = \emptyset$. Then $\pi_{\Lambda_x} : X \twoheadrightarrow X_{\Lambda_x} \subset \mathbb{P}^{n+1}$ is isomorphic near x and $x \notin D_{\Lambda_x}$ and $B_2 := (d-n-2)H - K_X$ is basepoint-free. Since $(d-1-i)H = K_X + (n+1-i)H + B_2$, by Kodaira vanishing theorem $H^i(\mathcal{O}_X(d-i-1)) = 0, \ i \geq 1$. □

Remark

$\mathrm{reg}_H(\mathcal{O}_X) \leq d-1$ for a smooth projective variety X.

Proposition(Mumford)
V is a basepoint-free subsystem in $H^0(\mathcal{O}_X(B_2))$. In particular,

$$H^i(\mathcal{O}_X(d-i-1)) = 0, \quad i \geq 1.$$

Proof. For $x \in X$, take a general linear space Λ_x such that $\Lambda_x \cap (T_x(X)^n \cup Cone(x, X)^{n+1}) = \emptyset$.Then $\pi_{\Lambda_x} : X \twoheadrightarrow X_{\Lambda_x} \subset \mathbb{P}^{n+1}$ is isomorphic near x and $x \notin D_{\Lambda_x}$ and $B_2 := (d-n-2)H - K_X$ is basepoint-free. Since $(d-1-i)H = K_X + (n+1-i)H + B_2$, by Kodaira vanishing theorem $H^i(\mathcal{O}_X(d-i-1)) = 0, \ i \geq 1.$ □

Remark

$\mathrm{reg}_H(\mathcal{O}_X) \leq d-1$ for a smooth projective variety X.

Proposition(Mumford)
V is a basepoint-free subsystem in $H^0(\mathcal{O}_X(B_2))$. In particular,

$$H^i(\mathcal{O}_X(d-i-1)) = 0, \ \ i \geq 1.$$

Proof. For $x \in X$, take a general linear space Λ_x such that $\Lambda_x \cap (T_x(X)^n \cup Cone(x, X)^{n+1}) = \emptyset$.Then $\pi_{\Lambda_x} : X \twoheadrightarrow X_{\Lambda_x} \subset \mathbb{P}^{n+1}$ is isomorphic near x and $x \notin D_{\Lambda_x}$ and $B_2 := (d-n-2)H - K_X$ is basepoint-free. Since $(d-1-i)H = K_X + (n+1-i)H + B_2$, by Kodaira vanishing theorem $H^i(\mathcal{O}_X(d-i-1)) = 0, \ i \geq 1$. □

Remark
$\mathrm{reg}_H(\mathcal{O}_X) \leq d-1$ for a smooth projective variety X.

Proposition (Mumford)
V is a basepoint-free subsystem in $H^0(\mathcal{O}_X(B_2))$. In particular,

$$H^i(\mathcal{O}_X(d-i-1))=0,\ \ i\geq 1.$$

Proof. For $x\in X$, take a general linear space Λ_x such that $\Lambda_x\cap(T_x(X)^n\cup Cone(x,X)^{n+1})=\emptyset$. Then $\pi_{\Lambda_x}: X\twoheadrightarrow X_{\Lambda_x}\subset\mathbb{P}^{n+1}$ is isomorphic near x and $x\notin D_{\Lambda_x}$ and $B_2:=(d-n-2)H-K_X$ is basepoint-free. Since $(d-1-i)H=K_X+(n+1-i)H+B_2$, by Kodaira vanishing theorem $H^i(\mathcal{O}_X(d-i-1))=0,\ i\geq 1.$ □

Remark

$\mathrm{reg}_H(\mathcal{O}_X)\leq d-1$ for a smooth projective variety X.

Remark

B.Ilic(1995) proved that B_2 is big unless X is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^{n-1}$. By Kawamata-Viehweg vanishing, we have

$$H^i(\mathcal{O}_X(d-2-i)) = 0, \quad i \geq 1.$$

Double point divisors from inner projections

Let $x_1, \ldots, x_{e-1} \in X$ be general points, and let $\Lambda := \langle x_1, \ldots, x_{e-1} \rangle$. Consider the inner projection at Λ and the blow-up at $x_1, \ldots, x_{e-1}$.

$$\begin{array}{ccccc} \widetilde{X} & \xrightarrow{\sigma} & X & \hookrightarrow & \mathbb{P}^{n+e} \\ & \searrow^{\widetilde{\pi}} & \downarrow^{\pi} & & \downarrow^{\pi_\Lambda} \\ & & \overline{X}_\Lambda & \hookrightarrow & \mathbb{P}^{n+1} \end{array}$$

Note that $\deg(\overline{X}_\Lambda) = \deg(X) - (e-1) = d - (e-1)$.

Remark

B.Ilic(1995) proved that B_2 is big unless X is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^{n-1}$.By Kawamata-Viehweg vanishing, we have

$$H^i(\mathcal{O}_X(d-2-i)) = 0, \quad i \geq 1.$$

Double point divisors from inner projections

Let $x_1, \ldots, x_{e-1} \in X$ be general points, and let $\Lambda := \langle x_1, \ldots, x_{e-1} \rangle$. Consider the inner projection at Λ and the blow-up at $x_1, \ldots, x_{e-1}$.

$$\begin{array}{ccccc} \widetilde{X} & \xrightarrow{\sigma} & X & \hookrightarrow & \mathbb{P}^{n+e} \\ & \searrow^{\widetilde{\pi}} & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_\Lambda} \\ & & \overline{X}_\Lambda & \hookrightarrow & \mathbb{P}^{n+1} \end{array}$$

Note that $\deg(\overline{X}_\Lambda) = \deg(X) - (e-1) = d - (e-1)$.

Remark

B.Ilic(1995) proved that B_2 is big unless X is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^{n-1}$.By Kawamata-Viehweg vanishing, we have

$$H^i(\mathcal{O}_X(d-2-i)) = 0, \quad i \geq 1.$$

Double point divisors from inner projections

Let $x_1, \ldots, x_{e-1} \in X$ be general points, and let $\Lambda := \langle x_1, \ldots, x_{e-1} \rangle$. Consider the inner projection at Λ and the blow-up at $x_1, \ldots, x_{e-1}$.

$$\begin{array}{ccccc} \widetilde{X} & \xrightarrow{\sigma} & X & \hookrightarrow & \mathbb{P}^{n+e} \\ & \searrow^{\widetilde{\pi}} & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_\Lambda} \\ & & \overline{X}_\Lambda & \hookrightarrow & \mathbb{P}^{n+1} \end{array}$$

Note that $\deg(\overline{X}_\Lambda) = \deg(X) - (e-1) = d - (e-1)$.

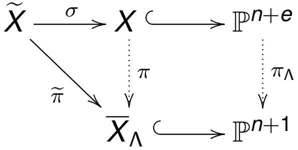
Remark

B.Ilic(1995) proved that B_2 is big unless X is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^{n-1}$.By Kawamata-Viehweg vanishing, we have

$$H^i(\mathcal{O}_X(d-2-i)) = 0, \quad i \geq 1.$$

Double point divisors from inner projections

Let $x_1, \ldots, x_{e-1} \in X$ be general points, and let $\Lambda := \langle x_1, \ldots, x_{e-1} \rangle$. Consider the inner projection at Λ and the blow-up at $x_1, \ldots, x_{e-1}$.



Note that $\deg(\overline{X}_\Lambda) = \deg(X) - (e-1) = d - (e-1)$.

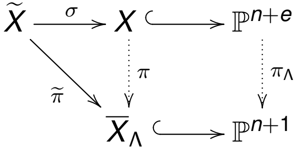
Remark

B.Ilic(1995) proved that B_2 is big unless X is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^{n-1}$.By Kawamata-Viehweg vanishing, we have

$$H^i(\mathcal{O}_X(d-2-i)) = 0, \quad i \geq 1.$$

Double point divisors from inner projections

Let $x_1, \ldots, x_{e-1} \in X$ be general points, and let $\Lambda := \langle x_1, \ldots, x_{e-1} \rangle$. Consider the inner projection at Λ and the blow-up at $x_1, \ldots, x_{e-1}$.



Note that $\deg(\overline{X}_\Lambda) = \deg(X) - (e-1) = d - (e-1)$.

Assume that X is neither a scroll over a curve nor a second Veronese surface. Then $\widetilde{\pi} : \widetilde{X} \twoheadrightarrow \overline{X}_\Lambda$ has no exceptional divisor(A. Noma, 2013). So, by the birational double point formula, we obtain an effective divisor $D(\widetilde{\pi})$ which is the non-isomorphic locus of $\widetilde{\pi}$. Then,

$$D(\pi) := \overline{\sigma(D(\widetilde{\pi})|_{\widetilde{X} \setminus E_1 \cup \cdots \cup E_{e-1}}}$$

is an effective divisor linearly equivalent to

$$B_{2,inn} := (d - n - e - 1)H - K_X,$$

which is called a double point divisor from inner projection.

Assume that X is neither a scroll over a curve nor a second Veronese surface. Then $\widetilde{\pi} : \widetilde{X} \twoheadrightarrow \overline{X}_\Lambda$ has no exceptional divisor(A. Noma, 2013).So, by the birational double point formula, we obtain an effective divisor $D(\widetilde{\pi})$ which is the non-isomorphic locus of $\widetilde{\pi}$.Then,

$$D(\pi) := \overline{\sigma(D(\widetilde{\pi})|_{\widetilde{X} \setminus E_1 \cup \cdots \cup E_{e-1}}}$$

is an effective divisor linearly equivalent to

$$B_{2,inn} := (d - n - e - 1)H - K_X,$$

which is called a double point divisor from inner projection.

Assume that X is neither a scroll over a curve nor a second Veronese surface. Then $\widetilde{\pi} : \widetilde{X} \twoheadrightarrow \overline{X}_\Lambda$ has no exceptional divisor(A. Noma, 2013).So, by the birational double point formula, we obtain an effective divisor $D(\widetilde{\pi})$ which is the non-isomorphic locus of $\widetilde{\pi}$.Then,

$$D(\pi) := \overline{\sigma(D(\widetilde{\pi})|_{\widetilde{X} \setminus E_1 \cup \cdots \cup E_{e-1}}}$$

is an effective divisor linearly equivalent to

$$B_{2,inn} := (d - n - e - 1)H - K_X,$$

which is called a double point divisor from inner projection.

Assume that X is neither a scroll over a curve nor a second Veronese surface. Then $\widetilde{\pi} : \widetilde{X} \twoheadrightarrow \overline{X}_\Lambda$ has no exceptional divisor(A. Noma, 2013).So, by the birational double point formula, we obtain an effective divisor $D(\widetilde{\pi})$ which is the non-isomorphic locus of $\widetilde{\pi}$.Then,

$$D(\pi) := \overline{\sigma(D(\widetilde{\pi})|_{\widetilde{X} \setminus E_1 \cup \cdots \cup E_{e-1}}}$$

is an effective divisor linearly equivalent to

$$B_{2,inn} := (d - n - e - 1)H - K_X,$$

which is called a double point divisor from inner projection.

If X is neither a scroll over a curve, a second Veronese surface, nor a Roth variety, then, $\mathcal{C}(X)$ is finite and we have the following:

Proposition (Noma)

$B_{2,inn}$ is semiample, i.e. some power of $B_{2,inn}$ is basepoint-free.

Proof: For any point $x \in X \setminus \mathcal{C}(X)$, by varying centers, we can take an inner projection $\pi\colon X \dashrightarrow \overline{X_\Lambda} \subset \mathbb{P}^{n+1}$ isomorphic at x. □

Note that b.p.f. $\Rightarrow$ semiample $\Rightarrow$ nef.

Corollary

$\mathrm{reg}_H(\mathcal{O}_X) \leq d - e$ if X is neither a scroll over a curve, a second Veronese surface, nor a Roth variety.

Proof: $(d-e-i)H = K_X + (n+1-i)H + B_{2,inn}$.

If X is neither a scroll over a curve, a second Veronese surface, nor a Roth variety, then, $\mathcal{C}(X)$ is finite and we have the following:

Proposition (Noma)

$B_{2,inn}$ is semiample, i.e. some power of $B_{2,inn}$ is basepoint-free.

Proof: For any point $x \in X \setminus \mathcal{C}(X)$, by varying centers, we can take an inner projection $\pi : X \dashrightarrow \overline{X}_\Lambda \subset \mathbb{P}^{n+1}$ isomorphic at x. □

Note that b.p.f. ⇒ semiample ⇒ nef.

Corollary

$\mathrm{reg}_H(\mathcal{O}_X) \le d - e$ if X is neither a scroll over a curve, a second Veronese surface, nor a Roth variety.

Proof: $(d-e-i)H = K_X + (n+1-i)H + B_{2,inn}$.

If X is neither a scroll over a curve, a second Veronese surface, nor a Roth variety, then, $\mathcal{C}(X)$ is finite and we have the following:

Proposition (Noma)

$B_{2,inn}$ is semiample, i.e. some power of $B_{2,inn}$ is basepoint-free.

Proof: For any point $x \in X \setminus \mathcal{C}(X)$, by varying centers, we can take an inner projection $\pi\colon X \dashrightarrow \overline{X}_\Lambda \subset \mathbb{P}^{n+1}$ isomorphic at x. $\square$

Note that b.p.f. $\Rightarrow$ semiample $\Rightarrow$ nef.

Corollary

$\mathrm{reg}_H(\mathcal{O}_X) \leq d - e$ if X is neither a scroll over a curve, a second Veronese surface, nor a Roth variety.

Proof: $(d - e - i)H = K_X + (n + 1 - i)H + B_{2,inn}$.

If X is neither a scroll over a curve, a second Veronese surface, nor a Roth variety, then, $\mathcal{C}(X)$ is finite and we have the following:

Proposition (Noma)

$B_{2,inn}$ is semiample, i.e. some power of $B_{2,inn}$ is basepoint-free.

Proof: For any point $x \in X \setminus \mathcal{C}(X)$, by varying centers, we can take an inner projection $\pi \colon X \dashrightarrow \overline{X}_\Lambda \subset \mathbb{P}^{n+1}$ isomorphic at x. □

Note that b.p.f. $\Rightarrow$ semiample $\Rightarrow$ nef.

Corollary

$\mathrm{reg}_H(\mathcal{O}_X) \leq d - e$ *if X is neither a scroll over a curve, a second Veronese surface, nor a Roth variety.*

Proof: $(d - e - i)H = K_X + (n + 1 - i)H + B_{2,inn}$.

If X is neither a scroll over a curve, a second Veronese surface, nor a Roth variety, then, $\mathcal{C}(X)$ is finite and we have the following:

Proposition (Noma)

$B_{2,inn}$ is semiample, i.e. some power of $B_{2,inn}$ is basepoint-free.

Proof: For any point $x \in X \setminus \mathcal{C}(X)$, by varying centers, we can take an inner projection $\pi : X \dashrightarrow \overline{X}_\Lambda \subset \mathbb{P}^{n+1}$ isomorphic at x. $\square$

Note that b.p.f. $\Rightarrow$ semiample $\Rightarrow$ nef.

Corollary

$\operatorname{reg}_H(\mathcal{O}_X) \leq d - e$ if X is neither a scroll over a curve, a second Veronese surface, nor a Roth variety.

Proof: $(d - e - i)H = K_X + (n + 1 - i)H + B_{2,inn}$.

Proposition

We have the following $\mathcal{O}_X$-regularity bound for smooth varieties;

- $X = v_2(\mathbb{P}^2) \subset \mathbb{P}^4$ *or in* $\mathbb{P}^5 \Rightarrow \mathrm{reg}(\mathcal{O}_X) = 1$;
- X : *Roth variety* $\Rightarrow \mathrm{reg}(\mathcal{O}_X) \leq d - e$;
- X : *a Scroll over a curve of genus g:*
  - $g = 0 \Rightarrow \mathrm{reg}(\mathcal{O}_X) = 1$;
  - $g = 1 \Rightarrow \mathrm{reg}(\mathcal{O}_X) = 2$;
  - $g \geq 2 \Rightarrow \mathrm{reg}(\mathcal{O}_X) \geq d - e - 2$.

Therefore, the second part of Eisenbud-Goto conjecture(i.e. $\mathrm{reg}(\mathcal{O}_X) \leq d - e$ is proved for smooth varieties. Thus,

$$\mathrm{reg}(R(X)) \leq d - e.$$

But, this part looks to be intrinsic without regard to the projective embedding of X.

Proposition

We have the following $\mathcal{O}_X$-regularity bound for smooth varieties;

- $X = \upsilon_2(\mathbb{P}^2) \subset \mathbb{P}^4$ *or in* $\mathbb{P}^5 \Rightarrow \mathrm{reg}(\mathcal{O}_X) = 1$;
- X : *Roth variety* $\Rightarrow \mathrm{reg}(\mathcal{O}_X) \leq d - e$;
- X : *a Scroll over a curve of genus g:*
  - $g = 0 \Rightarrow \mathrm{reg}(\mathcal{O}_X) = 1$;
  - $g = 1 \Rightarrow \mathrm{reg}(\mathcal{O}_X) = 2$;
  - $g \geq 2 \Rightarrow \mathrm{reg}(\mathcal{O}_X) \geq d - e - 2$.

Therefore, the second part of Eisenbud-Goto conjecture(i.e. $\mathrm{reg}(\mathcal{O}_X) \leq d - e$ is proved for smooth varieties. Thus,

$$\mathrm{reg}(R(X)) \leq d - e.$$

But, this part looks to be intrinsic without regard to the projective embedding of X.

Proposition

We have the following $\mathcal{O}_X$-regularity bound for smooth varieties;

- $X = \upsilon_2(\mathbb{P}^2) \subset \mathbb{P}^4$ *or in* $\mathbb{P}^5 \Rightarrow \mathrm{reg}(\mathcal{O}_X) = 1$;
- X : *Roth variety* $\Rightarrow \mathrm{reg}(\mathcal{O}_X) \leq d - e$;
- X : *a Scroll over a curve of genus g:*
  - $g = 0 \Rightarrow \mathrm{reg}(\mathcal{O}_X) = 1$;
  - $g = 1 \Rightarrow \mathrm{reg}(\mathcal{O}_X) = 2$;
  - $g \geq 2 \Rightarrow \mathrm{reg}(\mathcal{O}_X) \geq d - e - 2$.

Therefore, the second part of Eisenbud-Goto conjecture(i.e. $\mathrm{reg}(\mathcal{O}_X) \leq d - e$ is proved for smooth varieties. Thus,

$$\mathrm{reg}(R(X)) \leq d - e.$$

But, this part looks to be intrinsic without regard to the projective embedding of X.

Proposition

We have the following $\mathcal{O}_X$-regularity bound for smooth varieties;

- $X = \upsilon_2(\mathbb{P}^2) \subset \mathbb{P}^4$ *or in* $\mathbb{P}^5 \Rightarrow \mathrm{reg}(\mathcal{O}_X) = 1$;
- X : *Roth variety* $\Rightarrow \mathrm{reg}(\mathcal{O}_X) \leq d - e$;
- X : *a Scroll over a curve of genus g:*
  - $g = 0 \Rightarrow \mathrm{reg}(\mathcal{O}_X) = 1$;
  - $g = 1 \Rightarrow \mathrm{reg}(\mathcal{O}_X) = 2$;
  - $g \geq 2 \Rightarrow \mathrm{reg}(\mathcal{O}_X) \geq d - e - 2$.

Therefore, the second part of Eisenbud-Goto conjecture(i.e. $\mathrm{reg}(\mathcal{O}_X) \leq d - e$ is proved for smooth varieties. Thus,

$$\mathrm{reg}(R(X)) \leq d - e.$$

But, this part looks to be intrinsic without regard to the projective embedding of X.

Proposition

We have the following $\mathcal{O}_X$-regularity bound for smooth varieties;

- $X = \upsilon_2(\mathbb{P}^2) \subset \mathbb{P}^4$ *or in* $\mathbb{P}^5 \Rightarrow \mathrm{reg}(\mathcal{O}_X) = 1$;
- X : *Roth variety* $\Rightarrow \mathrm{reg}(\mathcal{O}_X) \leq d - e$;
- X : *a Scroll over a curve of genus g:*
    - $g = 0 \Rightarrow \mathrm{reg}(\mathcal{O}_X) = 1$;
    - $g = 1 \Rightarrow \mathrm{reg}(\mathcal{O}_X) = 2$;
    - $g \geq 2 \Rightarrow \mathrm{reg}(\mathcal{O}_X) \geq d - e - 2$.

Therefore, the second part of Eisenbud-Goto conjecture(i.e. $\mathrm{reg}(\mathcal{O}_X) \leq d - e$ is proved for smooth varieties. Thus,

$$\mathrm{reg}(R(X)) \leq d - e.$$

But, this part looks to be intrinsic without regard to the projective embedding of X.

Proposition

We have the following $\mathcal{O}_X$-regularity bound for smooth varieties;

- $X = \upsilon_2(\mathbb{P}^2) \subset \mathbb{P}^4$ *or in* $\mathbb{P}^5 \Rightarrow \mathrm{reg}(\mathcal{O}_X) = 1$;
- X : *Roth variety* $\Rightarrow \mathrm{reg}(\mathcal{O}_X) \leq d - e$;
- X : *a Scroll over a curve of genus* g:
  - $g = 0 \Rightarrow \mathrm{reg}(\mathcal{O}_X) = 1$;
  - $g = 1 \Rightarrow \mathrm{reg}(\mathcal{O}_X) = 2$;
  - $g \geq 2 \Rightarrow \mathrm{reg}(\mathcal{O}_X) \geq d - e - 2$.

Therefore, the second part of Eisenbud-Goto conjecture(i.e. $\mathrm{reg}(\mathcal{O}_X) \leq d - e$ is proved for smooth varieties. Thus,

$$\mathrm{reg}(R(X)) \leq d - e.$$

But, this part looks to be intrinsic without regard to the projective embedding of X.

Proposition

We have the following $\mathcal{O}_X$-regularity bound for smooth varieties;

- $X = \upsilon_2(\mathbb{P}^2) \subset \mathbb{P}^4$ *or in* $\mathbb{P}^5 \Rightarrow \mathrm{reg}(\mathcal{O}_X) = 1$;
- X : *Roth variety* $\Rightarrow \mathrm{reg}(\mathcal{O}_X) \leq d - e$;
- X : *a Scroll over a curve of genus g:*
  - $g = 0 \Rightarrow \mathrm{reg}(\mathcal{O}_X) = 1$;
  - $g = 1 \Rightarrow \mathrm{reg}(\mathcal{O}_X) = 2$;
  - $g \geq 2 \Rightarrow \mathrm{reg}(\mathcal{O}_X) \geq d - e - 2$.

Therefore, the second part of Eisenbud-Goto conjecture(i.e. $\mathrm{reg}(\mathcal{O}_X) \leq d - e$ is proved for smooth varieties. Thus,

$$\mathrm{reg}(R(X)) \leq d - e.$$

But, this part looks to be intrinsic without regard to the projective embedding of X.

Generalization to ND(1)-subscheme

- Our results can be generalized to ND(1)-subschemes using generic initial ideals in graded reverse lexicographic order.

Definition

A closed subscheme $X^n \subset \mathbb{P}^{n+e}$ is called ND(1)-subscheme if $X \cap \Lambda$ is nondegenerate for a general Λ of dimension $e \leq \dim \Lambda \leq n+e$.

- a ND(1)-subscheme is not necessarily to be *irreducible, reduced or equi-dimensional* in general.

Example (ND(1)-subschemes)

- Nondegenerate varieties;
- Connected in codimension 1 algebraic sets, i.e. X is equidimensional and each components are ordered such that $X_i \cap X_{i+1}$ is of codim 1 in X ;
- Algebraic sets whose one component is nondegenerate;

Generalization to ND(1)-subscheme

- Our results can be generalized to ND(1)-subschemes using generic initial ideals in graded reverse lexicographic order.

Definition

A closed subscheme $X^n \subset \mathbb{P}^{n+e}$ is called ND(1)-subscheme if $X \cap \Lambda$ is nondegenerate for a general Λ of dimension $e \leq \dim \Lambda \leq n+e$.

- a ND(1)-subscheme is not necessarily to be *irreducible, reduced or equi-dimensional* in general.

Example (ND(1)-subschemes)

- Nondegenerate varieties;
- Connected in codimension 1 algebraic sets, i.e. X is equidimensional and each components are ordered such that $X_i \cap X_{i+1}$ is of codim 1 in X ;
- Algebraic sets whose one component is nondegenerate;

Generalization to $\mathrm{ND}(1)$-subscheme

▶ Our results can be generalized to $\mathrm{ND}(1)$-subschemes using generic initial ideals in graded reverse lexicographic order.

Definition

A closed subscheme $X^n \subset \mathbb{P}^{n+e}$ is called $\mathrm{ND}(1)$-subscheme if $X \cap \Lambda$ is nondegenerate for a general Λ of dimension $e \leq \dim \Lambda \leq n+e$.

▶ a $\mathrm{ND}(1)$-subscheme is not necessarily to be *irreducible, reduced or equi-dimensional* in general.

Example ($\mathrm{ND}(1)$-subschemes)

- Nondegenerate varieties;
- Connected in codimension 1 algebraic sets, i.e. X is equidimensional and each components are ordered such that $X_i \cap X_{i+1}$ is of codim 1 in X ;
- Algebraic sets whose one component is nondegenerate;

Generalization to $\mathrm{ND}(1)$-subscheme

► Our results can be generalized to $\mathrm{ND}(1)$-subschemes using generic initial ideals in graded reverse lexicographic order.

Definition

A closed subscheme $X^n \subset \mathbb{P}^{n+e}$ is called $\mathrm{ND}(1)$-subscheme if $X \cap \Lambda$ is nondegenerate for a general Λ of dimension $e \leq \dim \Lambda \leq n+e$.

► a $\mathrm{ND}(1)$-subscheme is not necessarily to be *irreducible, reduced or equi-dimensional* in general.

Example ($\mathrm{ND}(1)$-subschemes)

- Nondegenerate varieties;
- Connected in codimension 1 algebraic sets, i.e. X is equidimensional and each components are ordered such that $X_i \cap X_{i+1}$ is of codim 1 in X ;
- Algebraic sets whose one component is nondegenerate;

Generalization to ND(1)-subscheme

▶ Our results can be generalized to ND(1)-subschemes using generic initial ideals in graded reverse lexicographic order.

Definition

A closed subscheme $X^n \subset \mathbb{P}^{n+e}$ is called ND(1)-subscheme if $X \cap \Lambda$ is nondegenerate for a general Λ of dimension $e \leq \dim \Lambda \leq n + e$.

▶ a ND(1)-subscheme is not necessarily to be *irreducible, reduced or equi-dimensional* in general.

Example (ND(1)-subschemes)

- Nondegenerate varieties;
- Connected in codimension 1 algebraic sets, i.e. X is equidimensional and each components are ordered such that $X_i \cap X_{i+1}$ is of codim 1 in X ;
- Algebraic sets whose one component is nondegenerate;

Generalization to $\mathrm{ND}(1)$-subscheme

▶ Our results can be generalized to $\mathrm{ND}(1)$-subschemes using generic initial ideals in graded reverse lexicographic order.

Definition

A closed subscheme $X^n \subset \mathbb{P}^{n+e}$ is called $\mathrm{ND}(1)$-subscheme if $X \cap \Lambda$ is nondegenerate for a general Λ of dimension $e \leq \dim \Lambda \leq n + e$.

▶ a $\mathrm{ND}(1)$-subscheme is not necessarily to be *irreducible, reduced or equi-dimensional* in general.

Example ($\mathrm{ND}(1)$-subschemes)

- Nondegenerate varieties;
- Connected in codimension 1 algebraic sets, i.e. X is equidimensional and each components are ordered such that $X_i \cap X_{i+1}$ is of codim 1 in X ;
- Algebraic sets whose one component is nondegenerate;

Non–ND(1) subschemes

▶ Skew lines in $\mathbb{P}^3$ is non-ND(1) and its Betti table is as follows:

	0	1	2	3
0	1	–	–	–
1	–	4	4	1

This is a counter-example for all the previous results.

▶ Two planes meeting at one point in $\mathbb{P}^4$ is also non-ND(1).

▶ For a ND(1) subscheme in $\mathbb{P}^{n+e}$, it is easy to show $\deg(X) \geq e+1$. So, we can define a minimal degree ND(1)-scheme.

▶ Take a diagram explaining the category of ND(1).

Non–$\mathrm{ND}(1)$ subschemes

▶ Skew lines in $\mathbb{P}^3$ is non-$\mathrm{ND}(1)$ and its Betti table is as follows:

	0	1	2	3
0	1	–	–	–
1	–	4	4	1

This is a counter-example for all the previous results.

▶ Two planes meeting at one point in $\mathbb{P}^4$ is also non-$\mathrm{ND}(1)$.

▶ For a $\mathrm{ND}(1)$ subscheme in $\mathbb{P}^{n+e}$, it is easy to show $\deg(X) \geq e+1$.
So, we can define a minimal degree $\mathrm{ND}(1)$-scheme.

▶ Take a diagram explaining the category of $\mathrm{ND}(1)$.

Non–ND(1) subschemes

▶ Skew lines in $\mathbb{P}^3$ is non-ND(1) and its Betti table is as follows:

	0	1	2	3
0	1	–	–	–
1	–	4	4	1

This is a counter-example for all the previous results.

▶ Two planes meeting at one point in $\mathbb{P}^4$ is also non-ND(1).

▶ For a ND(1) subscheme in $\mathbb{P}^{n+e}$, it is easy to show $\deg(X) \geq e+1$. So, we can define a minimal degree ND(1)-scheme.

▶ Take a diagram explaining the category of ND(1).

Non–$\mathrm{ND}(1)$ subschemes

▶ Skew lines in $\mathbb{P}^3$ is non-$\mathrm{ND}(1)$ and its Betti table is as follows:

	0	1	2	3
0	1	–	–	–
1	–	4	4	1

This is a counter-example for all the previous results.

▶ Two planes meeting at one point in $\mathbb{P}^4$ is also non-$\mathrm{ND}(1)$.

▶ For a $\mathrm{ND}(1)$ subscheme in $\mathbb{P}^{n+e}$, it is easy to show $\deg(X) \geq e+1$.

So, we can define a minimal degree $\mathrm{ND}(1)$-scheme.

▶ Take a diagram explaining the category of $\mathrm{ND}(1)$.

Non–$\mathrm{ND}(1)$ subschemes

▶ Skew lines in $\mathbb{P}^3$ is non-$\mathrm{ND}(1)$ and its Betti table is as follows:

	0	1	2	3
0	1	–	–	–
1	–	4	4	1

This is a counter-example for all the previous results.

▶ Two planes meeting at one point in $\mathbb{P}^4$ is also non-$\mathrm{ND}(1)$.

▶ For a $\mathrm{ND}(1)$ subscheme in $\mathbb{P}^{n+e}$, it is easy to show $\deg(X) \geq e+1$.
So, we can define a minimal degree $\mathrm{ND}(1)$-scheme.

▶ Take a diagram explaining the category of $\mathrm{ND}(1)$.

Non–$\mathrm{ND}(1)$ subschemes

▶ Skew lines in $\mathbb{P}^3$ is non-$\mathrm{ND}(1)$ and its Betti table is as follows:

	0	1	2	3
0	1	–	–	–
1	–	4	4	1

This is a counter-example for all the previous results.

▶ Two planes meeting at one point in $\mathbb{P}^4$ is also non-$\mathrm{ND}(1)$.

▶ For a $\mathrm{ND}(1)$ subscheme in $\mathbb{P}^{n+e}$, it is easy to show $\deg(X) \geq e+1$.
So, we can define a minimal degree $\mathrm{ND}(1)$-scheme.

▶ Take a diagram explaining the category of $\mathrm{ND}(1)$.

Theorem [Ahn, Han and K-, preprint]
$X^n \subset \mathbb{P}^{n+e}$: a $\mathrm{ND}(1)$ subscheme, defined over $K = \overline{K}$ of char $(K) = 0$. Then, we have the following upper bound:

$$\beta_{p,1}(X) \le p\binom{e+1}{p+1}, \quad p \ge 1$$

and the following are equivalent:

(a) X is a $\mathrm{ND}(1)$ subscheme of minimal degree;
(b) $h^0(\mathcal{I}_{X/\mathbb{P}^{n+e}}(2)) = \binom{e+1}{2}$;
(c) one $\beta_{p,1}(X)$ achieves the maximum for some $p \ge 1$;
(d) X has ACM 2-linear resolution.

Corollary $\beta_{p,1}(X) = 0, \quad p \ge e+1$ for any $\mathrm{ND}(1)$ subscheme X.

Theorem [Ahn, Han and K-, preprint]
$X^n \subset \mathbb{P}^{n+e}$: a ND(1) subscheme, defined over $K = \overline{K}$ of char $(K) = 0$.
Then, we have the following upper bound:

$$\beta_{p,1}(X) \leq p\binom{e+1}{p+1}, \quad p \geq 1$$

and the following are equivalent:

(a) X is a ND(1) subscheme of minimal degree;

(b) $h^0(\mathcal{I}_{X/\mathbb{P}^{n+e}}(2)) = \binom{e+1}{2}$;

(c) one $\beta_{p,1}(X)$ achieves the maximum for some $p \geq 1$;

(d) X has ACM 2-linear resolution.

Corollary $\beta_{p,1}(X) = 0, \quad p \geq e+1$ for any ND(1) subscheme X.

Theorem [Ahn, Han and K-, preprint]
$X^n \subset \mathbb{P}^{n+e}$: a $\mathrm{ND}(1)$ subscheme, defined over $K = \overline{K}$ of char $(K) = 0$.
Then, we have the following upper bound:

$$\beta_{p,1}(X) \leq p\binom{e+1}{p+1}, \quad p \geq 1$$

and the following are equivalent:

(a) X is a $\mathrm{ND}(1)$ subscheme of minimal degree;
(b) $h^0(\mathcal{I}_{X/\mathbb{P}^{n+e}}(2)) = \binom{e+1}{2}$;
(c) one $\beta_{p,1}(X)$ achieves the maximum for some $p \geq 1$;
(d) X has ACM 2-linear resolution.

Corollary $\beta_{p,1}(X) = 0, \quad p \geq e+1$ for any $\mathrm{ND}(1)$ subscheme X.

Theorem [Ahn, Han and K-, preprint]
$X^n \subset \mathbb{P}^{n+e}$: a $\mathrm{ND}(1)$ subscheme, defined over $K = \overline{K}$ of char $(K) = 0$.
Then, we have the following upper bound:

$$\beta_{p,1}(X) \leq p\binom{e+1}{p+1}, \quad p \geq 1$$

and the following are equivalent:

(a) X is a $\mathrm{ND}(1)$ subscheme of minimal degree;
(b) $h^0(\mathcal{I}_{X/\mathbb{P}^{n+e}}(2)) = \binom{e+1}{2}$;
(c) one $\beta_{p,1}(X)$ achieves the maximum for some $p \geq 1$;
(d) X has ACM 2-linear resolution.

Corollary $\beta_{p,1}(X) = 0, \quad p \geq e+1$ for any $\mathrm{ND}(1)$ subscheme X.

Theorem [Ahn, Han and K-, preprint]
$X^n \subset \mathbb{P}^{n+e}$: a $\mathrm{ND}(1)$ subscheme, defined over $K = \overline{K}$ of char $(K) = 0$.
Then, we have the following upper bound:

$$\beta_{p,1}(X) \leq p\binom{e+1}{p+1}, \quad p \geq 1$$

and the following are equivalent:

(a) X is a $\mathrm{ND}(1)$ subscheme of minimal degree;

(b) $h^0(\mathcal{I}_{X/\mathbb{P}^{n+e}}(2)) = \binom{e+1}{2}$;

(c) one $\beta_{p,1}(X)$ achieves the maximum for some $p \geq 1$;

(d) X has ACM 2-linear resolution.

Corollary $\beta_{p,1}(X) = 0, \quad p \geq e+1$ for any $\mathrm{ND}(1)$ subscheme X.

Ideas of a proof in case char$(K) = 0$

Generic initial ideal

For $g = (g_{ij}) \in GL_{n+1}(K)$, consider $g(I) = \{g \cdot f \mid f \in I\}$ where $g \cdot f = f(gx_0, gx_1, \cdots, gx_{n+e})$ and $gx_i = \sum_{0 \le k \le n+e} g_{ik} x_k$.

Then, due to Galligo and Bayer-Stillman, $\mathrm{in}_\tau(g(I))$ is constant for a general change g. We will call this the *generic initial ideal of* I w.r.t τ.

- $\mathrm{Gin}(I_X)$: the generic initial ideal of I_X in reverse lexicographic order. Then, the Hilberet functions of R/I_X and $R/\mathrm{Gin}(I_X)$ are the same.
  - $\mathrm{Gin}(I_X)$ is Borel-fixed, i.e. if $m \in \mathrm{Gin}(I_X)$ is divisible by x_j, then $\frac{x_i}{x_j} m \in \mathrm{Gin}(I_X)$ for $i < j$;
  - $\beta_{i,j}(R/I_X) \le \beta_{i,j}(R/\mathrm{Gin}(I_X))$ (Cancellation principle).
  - $\mathrm{reg}(I_X) = \mathrm{reg}(\mathrm{Gin}(I_X))$ =the maximal degree of generators of $\mathrm{Gin}(I_X)$;

Ideas of a proof in case char$(K) = 0$

Generic initial ideal

For $g = (g_{ij}) \in GL_{n+1}(K)$, consider $g(I) = \{g \cdot f \mid f \in I\}$ where $g \cdot f = f(gx_0, gx_1, \cdots, gx_{n+e})$ and $gx_i = \sum_{0 \le k \le n+e} g_{ik} x_k$.

Then, due to Galligo and Bayer-Stillman, $\mathrm{in}_\tau(g(I))$ is constant for a general change g. We will call this the *generic initial ideal of I* w.r.t τ.

▶ $\mathrm{Gin}(I_X)$: the generic initial ideal of I_X in reverse lexicographic order. Then, the Hilberet functions of R/I_X and $R/\mathrm{Gin}(I_X)$ are the same.

- $\mathrm{Gin}(I_X)$ is Borel-fixed, i.e. if $m \in \mathrm{Gin}(I_X)$ is divisible by x_j, then $\frac{x_i}{x_j} m \in \mathrm{Gin}(I_X)$ for $i < j$;
- $\beta_{i,j}(R/I_X) \le \beta_{i,j}(R/\mathrm{Gin}(I_X))$ (Cancellation principle).
- $\mathrm{reg}(I_X) = \mathrm{reg}(\mathrm{Gin}(I_X))$ =the maximal degree of generators of $\mathrm{Gin}(I_X)$;

Ideas of a proof in case char$(K) = 0$

Generic initial ideal

For $g = (g_{ij}) \in GL_{n+1}(K)$, consider $g(I) = \{g \cdot f \mid f \in I\}$ where $g \cdot f = f(gx_0, gx_1, \cdots, gx_{n+e})$ and $gx_i = \sum_{0 \le k \le n+e} g_{ik} x_k$.

Then, due to Galligo and Bayer-Stillman, $\mathrm{in}_\tau(g(I))$ is constant for a general change g. We will call this the *generic initial ideal of I* w.r.t τ.

▶ $\mathrm{Gin}(I_X)$: the generic initial ideal of I_X in reverse lexicographic order. Then, the Hilberet functions of R/I_X and $R/\mathrm{Gin}(I_X)$ are the same.

- $\mathrm{Gin}(I_X)$ is Borel-fixed, i.e. if $m \in \mathrm{Gin}(I_X)$ is divisible by x_j, then $\frac{x_i}{x_j} m \in \mathrm{Gin}(I_X)$ for $i < j$;
- $\beta_{i,j}(R/I_X) \le \beta_{i,j}(R/\mathrm{Gin}(I_X))$ (Cancellation principle).
- $\mathrm{reg}(I_X) = \mathrm{reg}(\mathrm{Gin}(I_X))$ =the maximal degree of generators of $\mathrm{Gin}(I_X)$;

Ideas of a proof in case $\mathrm{char}(K) = 0$

Generic initial ideal

For $g = (g_{ij}) \in GL_{n+1}(K)$, consider $g(I) = \{g \cdot f \mid f \in I\}$ where $g \cdot f = f(gx_0, gx_1, \cdots, gx_{n+e})$ and $gx_i = \sum_{0 \le k \le n+e} g_{ik} x_k$.

Then, due to Galligo and Bayer-Stillman, $\mathrm{in}_\tau(g(I))$ is constant for a general change g. We will call this the *generic initial ideal of I* w.r.t τ.

► $\mathrm{Gin}(I_X)$: the generic initial ideal of I_X in reverse lexicographic order. Then, the Hilberet functions of R/I_X and $R/\mathrm{Gin}(I_X)$ are the same.

- $\mathrm{Gin}(I_X)$ is Borel-fixed, i.e. if $m \in \mathrm{Gin}(I_X)$ is divisible by x_j, then $\frac{x_i}{x_j} m \in \mathrm{Gin}(I_X)$ for $i < j$;
- $\beta_{i,j}(R/I_X) \le \beta_{i,j}(R/\mathrm{Gin}(I_X))$ (Cancellation principle).
- $\mathrm{reg}(I_X) = \mathrm{reg}(\mathrm{Gin}(I_X))$ =the maximal degree of generators of $\mathrm{Gin}(I_X)$;

Ideas of a proof in case $\mathrm{char}(K) = 0$

Generic initial ideal

For $g = (g_{ij}) \in GL_{n+1}(K)$, consider $g(I) = \{g \cdot f \mid f \in I\}$ where $g \cdot f = f(gx_0, gx_1, \cdots, gx_{n+e})$ and $gx_i = \sum_{0 \le k \le n+e} g_{ik} x_k$.

Then, due to Galligo and Bayer-Stillman, $\mathrm{in}_\tau(g(I))$ is constant for a general change g. We will call this the *generic initial ideal of I* w.r.t τ.

► $\mathrm{Gin}(I_X)$: the generic initial ideal of I_X in reverse lexicographic order. Then, the Hilberet functions of R/I_X and $R/\mathrm{Gin}(I_X)$ are the same.

- $\mathrm{Gin}(I_X)$ is Borel-fixed, i.e. if $m \in \mathrm{Gin}(I_X)$ is divisible by x_j, then $\frac{x_i}{x_j} m \in \mathrm{Gin}(I_X)$ for $i < j$;
- $\beta_{i,j}(R/I_X) \le \beta_{i,j}(R/\mathrm{Gin}(I_X))$ (Cancellation principle).
- $\mathrm{reg}(I_X) = \mathrm{reg}(\mathrm{Gin}(I_X)) =$the maximal degree of generators of $\mathrm{Gin}(I_X)$;

Ideas of a proof in case $\mathrm{char}(K) = 0$

Generic initial ideal

For $g = (g_{ij}) \in GL_{n+1}(K)$, consider $g(I) = \{g \cdot f \mid f \in I\}$ where $g \cdot f = f(gx_0, gx_1, \cdots, gx_{n+e})$ and $gx_i = \sum_{0 \le k \le n+e} g_{ik} x_k$.

Then, due to Galligo and Bayer-Stillman, $\mathrm{in}_\tau(g(I))$ is constant for a general change g. We will call this the *generic initial ideal of I* w.r.t τ.

▶ $\mathrm{Gin}(I_X)$: the generic initial ideal of I_X in reverse lexicographic order. Then, the Hilberet functions of R/I_X and $R/\mathrm{Gin}(I_X)$ are the same.

- $\mathrm{Gin}(I_X)$ is Borel-fixed, i.e. if $m \in \mathrm{Gin}(I_X)$ is divisible by x_j, then $\frac{x_i}{x_j} m \in \mathrm{Gin}(I_X)$ for $i < j$;
- $\beta_{i,j}(R/I_X) \le \beta_{i,j}(R/\mathrm{Gin}(I_X))$ (Cancellation principle).
- $\mathrm{reg}(I_X) = \mathrm{reg}(\mathrm{Gin}(I_X))$ =the maximal degree of generators of $\mathrm{Gin}(I_X)$;

- $\mathrm{Gin}(I_{X\cap H}) = (\mathrm{Gin}(I_X)|_{x_{n+e}\to 0})|_{x_{n+e-1}\to 1}$ for a general hyperplane H (Bayer-Stillman or M. Green);
(This is the main reason we use the reverse lexicographic order).

- For a generic initial ideal $\mathrm{Gin}(I_X)$, we can compute the graded Betti number $\beta_{i,j}(R/\mathrm{Gin}(I_X))$ by the combinatorial method due to the Eliahou-Kervaire Theorem.
  - $\beta_{i,j}(R/\mathrm{Gin}(I_X)) = \sum_{T\in\mathcal{G}(\mathrm{Gin}(I_X))_{j+1}} \binom{\max T}{i-1}$ (Eliahou-Kervaire), where
    - $\max T := \max\{t \mid x_t \text{ divides } T\}$;
    - $\mathcal{G}(\mathrm{Gin}(I_X))_m$: the set of minimal generators of degree m.

Recall again that

$$\beta_{i,j}(R/I_X) \leq \beta_{i,j}(R/\mathrm{Gin}(I_X)).$$

- $\mathrm{Gin}(I_{X\cap H}) = (\mathrm{Gin}(I_X)|_{x_{n+e}\to 0})|_{x_{n+e-1}\to 1}$ for a general hyperplane H (Bayer-Stillman or M. Green);
(This is the main reason we use the reverse lexicographic order).

- For a generic initial ideal $\mathrm{Gin}(I_X)$, we can compute the graded Betti number $\beta_{i,j}(R/\mathrm{Gin}(I_X))$ by the combinatorial method due to the Eliahou-Kervaire Theorem.
  - $\beta_{i,j}(R/\mathrm{Gin}(I_X)) = \sum_{T\in\mathcal{G}(\mathrm{Gin}(I_X))_{j+1}} \binom{\max T}{i-1}$ (Eliahou-Kervaire), where
    - $\max T := \max\{t \mid x_t \text{ divides } T\}$;
    - $\mathcal{G}(\mathrm{Gin}(I_X))_m$: the set of minimal generators of degree m.

Recall again that

$$\beta_{i,j}(R/I_X) \leq \beta_{i,j}(R/\mathrm{Gin}(I_X)).$$

▶ $\mathrm{Gin}(I_{X\cap H}) = (\mathrm{Gin}(I_X)|_{x_{n+e}\to 0})|_{x_{n+e-1}\to 1}$ for a general hyperplane H (Bayer-Stillman or M. Green);
(This is the main reason we use the reverse lexicographic order).

▶ For a generic initial ideal $\mathrm{Gin}(I_X)$, we can compute the graded Betti number $\beta_{i,j}(R/\mathrm{Gin}(I_X))$ by the combinatorial method due to the Eliahou-Kervaire Theorem.

- $\beta_{i,j}(R/\mathrm{Gin}(I_X)) = \sum_{T\in\mathcal{G}(\mathrm{Gin}(I_X))_{j+1}} \binom{\max T}{i-1}$ (Eliahou-Kervaire), where
  - ▶ $\max T := \max\{t \mid x_t \text{ divides } T\}$;
  - ▶ $\mathcal{G}(\mathrm{Gin}(I_X))_m$: the set of minimal generators of degree m.

Recall again that

$$\beta_{i,j}(R/I_X) \le \beta_{i,j}(R/\mathrm{Gin}(I_X)).$$

▶ $\mathrm{Gin}(I_{X\cap H}) = (\mathrm{Gin}(I_X)|_{x_{n+e}\to 0})|_{x_{n+e-1}\to 1}$ for a general hyperplane H (Bayer-Stillman or M. Green);
(This is the main reason we use the reverse lexicographic order).

▶ For a generic initial ideal $\mathrm{Gin}(I_X)$, we can compute the graded Betti number $\beta_{i,j}(R/\mathrm{Gin}(I_X))$ by the combinatorial method due to the Eliahou-Kervaire Theorem.

- $\beta_{i,j}(R/\mathrm{Gin}(I_X)) = \sum_{T\in\mathcal{G}(\mathrm{Gin}(I_X))_{j+1}} \binom{\max T}{i-1}$ (Eliahou-Kervaire), where
  - ▶ $\max T := \max\{t \mid x_t \text{ divides } T\}$;
  - ▶ $\mathcal{G}(\mathrm{Gin}(I_X))_m$: the set of minimal generators of degree m.

Recall again that

$$\beta_{i,j}(R/I_X) \leq \beta_{i,j}(R/\mathrm{Gin}(I_X)).$$

▶ $\mathrm{Gin}(I_{X\cap H}) = (\mathrm{Gin}(I_X)|_{x_{n+e}\to 0})|_{x_{n+e-1}\to 1}$ for a general hyperplane H (Bayer-Stillman or M. Green);
(This is the main reason we use the reverse lexicographic order).

▶ For a generic initial ideal $\mathrm{Gin}(I_X)$, we can compute the graded Betti number $\beta_{i,j}(R/\mathrm{Gin}(I_X))$ by the combinatorial method due to the Eliahou-Kervaire Theorem.

- $\beta_{i,j}(R/\mathrm{Gin}(I_X)) = \sum_{T\in\mathcal{G}(\mathrm{Gin}(I_X))_{j+1}} \binom{\max T}{i-1}$ (Eliahou-Kervaire), where
  - ▶ $\max T := \max\{t \mid x_t \text{ divides } T\}$;
  - ▶ $\mathcal{G}(\mathrm{Gin}(I_X))_m$: the set of minimal generators of degree m.

Recall again that

$$\beta_{i,j}(R/I_X) \leq \beta_{i,j}(R/\mathrm{Gin}(I_X)).$$

- [Important fact from ND(1)-property]

$$x_i x_j \notin \mathrm{Gin}(I_X), 0 \le i \le e-1, e \le j \le n+e$$

by the non-degenerate condition and Bayer-Stillman theorem $\mathrm{Gin}(I_{X\cap H}) = (\mathrm{Gin}(I_X)|_{x_{n+e}\to 0})|_{x_{n+e-1}\to 1}$. Therefore,

- By Eliahou-Kervaire, $\beta_{i,1}(R/\mathrm{Gin}(I_X)) = 0$ for $i > e$.
- Similarly, $\mathrm{Gin}(I_X) \subset (x_0, x_1, \ldots, x_{e-1})^2$ and
- $\beta_{i,1}(R/I_X) \le \beta_{i,1}(R/\mathrm{Gin}(I_X)) \le \beta_{i,1}(R/(x_0, \ldots, x_{e-1})^2) = i\binom{e+1}{i+1}$.

Note that the equality holds for some $i \ge 1$ if and only if $\mathrm{Gin}(I_X) = (x_0, x_1, \ldots, x_{e-1})^2$ if and only if $\deg(X) = e+1$. □

- [Important fact from $\mathrm{ND}(1)$-property]

$$x_i x_j \notin \mathrm{Gin}(I_X), 0 \le i \le e-1, e \le j \le n+e$$

by the non-degenerate condition and Bayer-Stillman theorem $\mathrm{Gin}(I_{X\cap H}) = (\mathrm{Gin}(I_X)|_{x_{n+e}\to 0})|_{x_{n+e-1}\to 1}$. Therefore,

- By Eliahou-Kervaire, $\beta_{i,1}(R/\mathrm{Gin}(I_X)) = 0$ for $i > e$.
- Similarly, $\mathrm{Gin}(I_X) \subset (x_0, x_1, \dots, x_{e-1})^2$ and
- $\beta_{i,1}(R/I_X) \le \beta_{i,1}(R/\mathrm{Gin}(I_X)) \le \beta_{i,1}(R/(x_0, \dots, x_{e-1})^2) = i\binom{e+1}{i+1}$.

Note that the equality holds for some $i \ge 1$ if and only if $\mathrm{Gin}(I_X) = (x_0, x_1, \dots, x_{e-1})^2$ if and only if $\deg(X) = e+1$. □

- [Important fact from ND(1)-property]

$$x_i x_j \notin \mathrm{Gin}(I_X), 0 \le i \le e-1, e \le j \le n+e$$

by the non-degenerate condition and Bayer-Stillman theorem $\mathrm{Gin}(I_{X\cap H}) = (\mathrm{Gin}(I_X)|_{x_{n+e}\to 0})|_{x_{n+e-1}\to 1}$. Therefore,

- By Eliahou-Kervaire, $\beta_{i,1}(R/\mathrm{Gin}(I_X)) = 0$ for $i > e$.
- Similarly, $\mathrm{Gin}(I_X) \subset (x_0, x_1, \dots, x_{e-1})^2$ and
- $\beta_{i,1}(R/I_X) \le \beta_{i,1}(R/\mathrm{Gin}(I_X)) \le \beta_{i,1}(R/(x_0, \dots, x_{e-1})^2) = i\binom{e+1}{i+1}$.

Note that the equality holds for some $i \ge 1$ if and only if $\mathrm{Gin}(I_X) = (x_0, x_1, \dots, x_{e-1})^2$ if and only if $\deg(X) = e+1$. □

- [Important fact from ND(1)-property]

$$x_i x_j \notin \mathrm{Gin}(I_X), 0 \le i \le e-1, e \le j \le n+e$$

by the non-degenerate condition and Bayer-Stillman theorem $\mathrm{Gin}(I_{X\cap H}) = (\mathrm{Gin}(I_X)|_{x_{n+e}\to 0})|_{x_{n+e-1}\to 1}$. Therefore,

- By Eliahou-Kervaire, $\beta_{i,1}(R/\mathrm{Gin}(I_X)) = 0$ for $i > e$.
- Similarly, $\mathrm{Gin}(I_X) \subset (x_0, x_1, \ldots, x_{e-1})^2$ and
- $\beta_{i,1}(R/I_X) \le \beta_{i,1}(R/\mathrm{Gin}(I_X)) \le \beta_{i,1}(R/(x_0,\ldots,x_{e-1})^2) = i\binom{e+1}{i+1}$.

Note that the equality holds for some $i \ge 1$ if and only if $\mathrm{Gin}(I_X) = (x_0, x_1, \ldots, x_{e-1})^2$ if and only if $\deg(X) = e+1$. □

- [Important fact from ND(1)-property]

$$x_i x_j \notin \mathrm{Gin}(I_X), 0 \le i \le e-1, e \le j \le n+e$$

by the non-degenerate condition and Bayer-Stillman theorem $\mathrm{Gin}(I_{X\cap H}) = (\mathrm{Gin}(I_X)|_{x_{n+e}\to 0})|_{x_{n+e-1}\to 1}$. Therefore,

- By Eliahou-Kervaire, $\beta_{i,1}(R/\mathrm{Gin}(I_X)) = 0$ for $i > e$.
- Similarly, $\mathrm{Gin}(I_X) \subset (x_0, x_1, \dots, x_{e-1})^2$ and
- $\beta_{i,1}(R/I_X) \le \beta_{i,1}(R/\mathrm{Gin}(I_X)) \le \beta_{i,1}(R/(x_0, \dots, x_{e-1})^2) = i\binom{e+1}{i+1}$.

Note that the equality holds for some $i \ge 1$ if and only if $\mathrm{Gin}(I_X) = (x_0, x_1, \dots, x_{e-1})^2$ if and only if $\deg(X) = e+1$. □

Our problems on syzygies on cubic generators

Let $X^n \subset \mathbb{P}^{n+e}$ be any variety over K. Suppose that $(I_X)_2 = 0$. Then,

(a) What can we say about upper bounds for $\beta_{p,2}(X)$ for any $p \geq 2$ and $K_{p,2}$ Theorem (i.e., $\beta_{p,2}(X) = 0$ for $p > e$ under some conditions.

(b) Classify or characterize (near) boundary cases for the upper bounds of $\beta_{p,2}(X)$.

Definition

A closed subscheme $X^n \subset \mathbb{P}^{n+e}$ is called ND(2)-subscheme if for a general Λ of dimension $e \leq \dim \Lambda \leq n+e$, $X \cap \Lambda$ is not contained in a quadric.

Our problems on syzygies on cubic generators

Let $X^n \subset \mathbb{P}^{n+e}$ be any variety over K. Suppose that $(I_X)_2 = 0$. Then,

(a) What can we say about upper bounds for $\beta_{p,2}(X)$ for any $p \geq 2$ and $K_{p,2}$ Theorem (i.e., $\beta_{p,2}(X) = 0$ for $p > e$ under some conditions.

(b) Classify or characterize (near) boundary cases for the upper bounds of $\beta_{p,2}(X)$.

Definition

A closed subscheme $X^n \subset \mathbb{P}^{n+e}$ is called ND(2)-subscheme if for a general Λ of dimension $e \leq \dim \Lambda \leq n+e$, $X \cap \Lambda$ is not contained in a quadric.

Our problems on syzygies on cubic generators

Let $X^n \subset \mathbb{P}^{n+e}$ be any variety over K. Suppose that $(I_X)_2 = 0$. Then,

(a) What can we say about upper bounds for $\beta_{p,2}(X)$ for any $p \geq 2$ and $K_{p,2}$ Theorem (i.e., $\beta_{p,2}(X) = 0$ for $p > e$ under some conditions.

(b) Classify or characterize (near) boundary cases for the upper bounds of $\beta_{p,2}(X)$.

Definition

A closed subscheme $X^n \subset \mathbb{P}^{n+e}$ is called ND(2)-subscheme if for a general Λ of dimension $e \leq \dim \Lambda \leq n+e$, $X \cap \Lambda$ is not contained in a quadric.

Our problems on syzygies on cubic generators

Let $X^n \subset \mathbb{P}^{n+e}$ be any variety over K. Suppose that $(I_X)_2 = 0$. Then,

(a) What can we say about upper bounds for $\beta_{p,2}(X)$ for any $p \geq 2$ and $K_{p,2}$ Theorem (i.e., $\beta_{p,2}(X) = 0$ for $p > e$ under some conditions.

(b) Classify or characterize (near) boundary cases for the upper bounds of $\beta_{p,2}(X)$.

Definition

A closed subscheme $X^n \subset \mathbb{P}^{n+e}$ is called $\mathrm{ND}(2)$-subscheme if for a general Λ of dimension $e \leq \dim \Lambda \leq n+e$, $X \cap \Lambda$ is not contained in a quadric.

Our problems on syzygies on cubic generators

Let $X^n \subset \mathbb{P}^{n+e}$ be any variety over K. Suppose that $(I_X)_2 = 0$. Then,

(a) What can we say about upper bounds for $\beta_{p,2}(X)$ for any $p \geq 2$ and $K_{p,2}$ Theorem (i.e., $\beta_{p,2}(X) = 0$ for $p > e$ under some conditions.

(b) Classify or characterize (near) boundary cases for the upper bounds of $\beta_{p,2}(X)$.

Definition

A closed subscheme $X^n \subset \mathbb{P}^{n+e}$ is called $\mathrm{ND}(2)$-subscheme if for a general Λ of dimension $e \leq \dim \Lambda \leq n+e$, $X \cap \Lambda$ is not contained in a quadric.

Example (ND(2)-subschemes)

- Nondegenerate linearly normal curve;
- Algebraic sets whose one component is a ND(2)-subscheme.

Consider the following Betti table starting from cubic generators for a ND(2)-subscheme X:

	0	1	2	3	$\cdots$	$i-1$	i	$i+1$	$\cdots$	$\triangle$
0	1	–	–	–	$\cdots$	–	–	–	$\cdots$	–
1	–	–	–	–	$\cdots$	–	–	–	$\cdots$	–
2	–	$\beta_{1,2}$	$\beta_{2,2}$	$\beta_{3,2}$	$\cdots$	$\beta_{i-1,2}$	$\beta_{i,2}$	$\beta_{i+1,2}$	$\cdots$	$\beta_{\triangle,2}$
$\vdots$	–	–	$\cdots$	–	$\ddots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\ddots$
j	–	$\beta_{1,j}$	$\beta_{2,j}$	$\beta_{3,j}$	$\cdots$	$\beta_{i-1,j}$	$\beta_{i,j}$	$\beta_{i+1,j}$	$\cdots$	$\beta_{\triangle,j}$

▶ It is easy to show $\deg(X) \geq \binom{e+2}{2}$ for a ND(2)-scheme X in $\mathbb{P}^{n+e}$,.

Example ($\mathrm{ND}(2)$-subschemes)

- Nondegenerate linearly normal curve;
- Algebraic sets whose one component is a $\mathrm{ND}(2)$-subscheme.

Consider the following Betti table starting from cubic generators for a $\mathrm{ND}(2)$-subscheme X:

	0	1	2	3	$\cdots$	$i-1$	i	$i+1$	$\cdots$	$\triangle$
0	1	–	–	–	$\cdots$	–	–	–	$\cdots$	–
1	–	–	–	–	$\cdots$	–	–	–	$\cdots$	–
2	–	$\beta_{1,2}$	$\beta_{2,2}$	$\beta_{3,2}$	$\cdots$	$\beta_{i-1,2}$	$\beta_{i,2}$	$\beta_{i+1,2}$	$\cdots$	$\beta_{\triangle,2}$
$\vdots$	–	–	$\ldots$	–	$\ddots$	$\ldots$	$\vdots$	$\vdots$	$\ldots$	$\ddots$
j	–	$\beta_{1,j}$	$\beta_{2,j}$	$\beta_{3,j}$	$\cdots$	$\beta_{i-1,j}$	$\beta_{i,j}$	$\beta_{i+1,j}$	$\cdots$	$\beta_{\triangle,j}$

▶ It is easy to show $\deg(X) \geq \binom{e+2}{2}$ for a $\mathrm{ND}(2)$-scheme X in $\mathbb{P}^{n+e}$,.

Example (ND(2)-subschemes)

- Nondegenerate linearly normal curve;
- Algebraic sets whose one component is a ND(2)-subscheme.

Consider the following Betti table starting from cubic generators for a ND(2)-subscheme X:

	0	1	2	3	$\cdots$	$i-1$	i	$i+1$	$\cdots$	$\triangle$
0	1	–	–	–	$\cdots$	–	–	–	$\cdots$	–
1	–	–	–	–	$\cdots$	–	–	–	$\cdots$	–
2	–	$\beta_{1,2}$	$\beta_{2,2}$	$\beta_{3,2}$	$\cdots$	$\beta_{i-1,2}$	$\beta_{i,2}$	$\beta_{i+1,2}$	$\cdots$	$\beta_{\triangle,2}$
$\vdots$	–	–	$\dots$	–	$\ddots$	$\dots$	$\vdots$	$\vdots$	$\dots$	$\ddots$
j	–	$\beta_{1,j}$	$\beta_{2,j}$	$\beta_{3,j}$	$\cdots$	$\beta_{i-1,j}$	$\beta_{i,j}$	$\beta_{i+1,j}$	$\cdots$	$\beta_{\triangle,j}$

- It is easy to show $\deg(X) \geq \binom{e+2}{2}$ for a ND(2)-scheme X in $\mathbb{P}^{n+e}$,.

Upper bound of $\beta_{p,2}(X)$ and $K_{p,2}$ Theorem

Theorem (Ahn, Han and K-, preprint)

Suppose that $X^n \subset \mathbb{P}^{n+e}$ is a $\mathrm{ND}(2)$ subscheme, defined over $K = \overline{K}$ of char $(K) = 0$. Then,

- $\binom{e+2}{2} \leq \deg(X)$ *and* $h^0(\mathcal{I}_X(3)) \leq \binom{e+2}{3}$.
- *In general,* $\beta_{p,2}(X) \leq \binom{p+1}{2}\binom{e+2}{p+2}$ *for* $p \geq 1$.
- *For the extremal cases, the following are equivalent:*
  - (a) $\deg(X) = \binom{e+2}{2}$;
  - (b) $h^0(\mathcal{I}_X(3)) = \binom{e+2}{3}$;
  - (c) *one of* $\beta_{p,2}(X)$ *attains "=" for* $1 \leq p \leq e$;
  - (d) I_X *has ACM* 3*-linear resolution.*

This also gives a natural $K_{p,2}$ theorem generalizing $K_{p,1}$-theorem because $\beta_{p,2}(X) = 0$ for $p > e$.

Upper bound of $\beta_{p,2}(X)$ and $K_{p,2}$ Theorem

Theorem (Ahn, Han and K-, preprint)

Suppose that $X^n \subset \mathbb{P}^{n+e}$ is a $\mathrm{ND}(2)$ subscheme, defined over $K = \overline{K}$ of char $(K) = 0$. Then,

- $\binom{e+2}{2} \leq \deg(X)$ *and* $h^0(\mathcal{I}_X(3)) \leq \binom{e+2}{3}$.
- *In general,* $\beta_{p,2}(X) \leq \binom{p+1}{2}\binom{e+2}{p+2}$ *for* $p \geq 1$.
- *For the extremal cases, the following are equivalent:*
  - (a) $\deg(X) = \binom{e+2}{2}$;
  - (b) $h^0(\mathcal{I}_X(3)) = \binom{e+2}{3}$;
  - (c) *one of* $\beta_{p,2}(X)$ *attains "=" for* $1 \leq p \leq e$;
  - (d) I_X *has ACM* 3*-linear resolution.*

This also gives a natural $K_{p,2}$ theorem generalizing $K_{p,1}$-theorem because $\beta_{p,2}(X) = 0$ for $p > e$.

Upper bound of $\beta_{p,2}(X)$ and $K_{p,2}$ Theorem

Theorem (Ahn, Han and K-, preprint)

Suppose that $X^n \subset \mathbb{P}^{n+e}$ is a $\mathrm{ND}(2)$ subscheme, defined over $K = \overline{K}$ of char $(K) = 0$. Then,

- $\binom{e+2}{2} \leq \deg(X)$ *and* $h^0(\mathcal{I}_X(3)) \leq \binom{e+2}{3}$.
- *In general,* $\beta_{p,2}(X) \leq \binom{p+1}{2}\binom{e+2}{p+2}$ *for* $p \geq 1$.
- *For the extremal cases, the following are equivalent:*
  - (a) $\deg(X) = \binom{e+2}{2}$;
  - (b) $h^0(\mathcal{I}_X(3)) = \binom{e+2}{3}$;
  - (c) *one of* $\beta_{p,2}(X)$ *attains "=" for* $1 \leq p \leq e$;
  - (d) I_X *has ACM* 3*-linear resolution.*

This also gives a natural $K_{p,2}$ theorem generalizing $K_{p,1}$-theorem because $\beta_{p,2}(X) = 0$ for $p > e$.

ACM varieties with 2-linear or 3 linear resolutions

- For $\mathrm{ND}(1)$-varieties, $\deg(X) = e + 1 \Leftrightarrow X$ is 2-linear ACM.
  This is a minimal degree variety in a category of $\mathrm{ND}(1)$-varieties.

	0	1	2	3	$\cdots$	i	$\cdots$	$e-1$	e
0	1	–	–	–	$\cdots$	–	$\cdots$	–	–
1	–	$\beta_{1,1}$	$\beta_{2,1}$	$\beta_{3,1}$	$\cdots$	$\beta_{i,1}$	$\cdots$	$\beta_{e-1,1}$	$\beta_{e,1}$

- For $\mathrm{ND}(2)$-varieties, $\deg(X) = \binom{e+2}{2} \Leftrightarrow X$ is 3-linear ACM.
  This has a minimal degree in a category of $\mathrm{ND}(2)$-varieties.

	0	1	2	3	$\cdots$	i	$\cdots$	$e-1$	e
0	1	–	–	–	$\cdots$	–	$\cdots$	–	–
1	–	–	–	–	$\cdots$	–	$\cdots$	–	–
2	–	$\beta_{1,2}$	$\beta_{2,2}$	$\beta_{3,2}$	$\cdots$	$\beta_{i,2}$	$\cdots$	$\beta_{e-1,2}$	$\beta_{e,2}$

ACM varieties with 2-linear or 3 linear resolutions

- For $\mathrm{ND}(1)$-varieties, $\deg(X) = e + 1 \Leftrightarrow X$ is 2-linear ACM.
  This is a minimal degree variety in a category of $\mathrm{ND}(1)$-varieties.

	0	1	2	3	$\cdots$	i	$\cdots$	$e-1$	e
0	1	–	–	–	$\cdots$	–	$\cdots$	–	–
1	–	$\beta_{1,1}$	$\beta_{2,1}$	$\beta_{3,1}$	$\cdots$	$\beta_{i,1}$	$\cdots$	$\beta_{e-1,1}$	$\beta_{e,1}$

- For $\mathrm{ND}(2)$-varieties, $\deg(X) = \binom{e+2}{2} \Leftrightarrow X$ is 3-linear ACM.
  This has a minimal degree in a category of $\mathrm{ND}(2)$-varieties.

	0	1	2	3	$\cdots$	i	$\cdots$	$e-1$	e
0	1	–	–	–	$\cdots$	–	$\cdots$	–	–
1	–	–	–	–	$\cdots$	–	$\cdots$	–	–
2	–	$\beta_{1,2}$	$\beta_{2,2}$	$\beta_{3,2}$	$\cdots$	$\beta_{i,2}$	$\cdots$	$\beta_{e-1,2}$	$\beta_{e,2}$

ACM varieties with 2-linear or 3 linear resolutions

- For $\mathrm{ND}(1)$-varieties, $\deg(X) = e + 1 \Leftrightarrow X$ is 2-linear ACM. This is a minimal degree variety in a category of $\mathrm{ND}(1)$-varieties.

	0	1	2	3	$\cdots$	i	$\cdots$	$e-1$	e
0	1	–	–	–	$\cdots$	–	$\cdots$	–	–
1	–	$\beta_{1,1}$	$\beta_{2,1}$	$\beta_{3,1}$	$\cdots$	$\beta_{i,1}$	$\cdots$	$\beta_{e-1,1}$	$\beta_{e,1}$

- For $\mathrm{ND}(2)$-varieties, $\deg(X) = \binom{e+2}{2} \Leftrightarrow X$ is 3-linear ACM. This has a minimal degree in a category of $\mathrm{ND}(2)$-varieties.

	0	1	2	3	$\cdots$	i	$\cdots$	$e-1$	e
0	1	–	–	–	$\cdots$	–	$\cdots$	–	–
1	–	–	–	–	$\cdots$	–	$\cdots$	–	–
2	–	$\beta_{1,2}$	$\beta_{2,2}$	$\beta_{3,2}$	$\cdots$	$\beta_{i,2}$	$\cdots$	$\beta_{e-1,2}$	$\beta_{e,2}$

- Main question at this point is to give geometric classification/or characterization of ACM varieties with 3-linear resolution?

[Examples of varieties having ACM 3-linear resolution]

(a) Hypercubic ($e = 1$);

(b) 3-minors of 4×4 generic symmetric matrix (i.e. $\mathrm{Sec}(v_2(\mathbb{P}^3)) \subset \mathbb{P}^9$);

(c) 3-minors of $3 \times (e + 2)$ sufficiently generic matrices (e.g. $Sec(RNS)$).

- Main question at this point is to give geometric classification/or characterization of ACM varieties with 3-linear resolution?

[Examples of varieties having ACM 3-linear resolution]

(a) Hypercubic ($e = 1$);

(b) 3-minors of 4×4 generic symmetric matrix (i.e. $\mathrm{Sec}(v_2(\mathbb{P}^3)) \subset \mathbb{P}^9$);

(c) 3-minors of $3 \times (e + 2)$ sufficiently generic matrices (e.g. $Sec(RNS)$).

▶ Main question at this point is to give geometric classification/or characterization of ACM varieties with 3-linear resolution?

[Examples of varieties having ACM 3-linear resolution]

(a) Hypercubic ($e = 1$);

(b) 3-minors of 4×4 generic symmetric matrix (i.e. $\mathrm{Sec}(v_2(\mathbb{P}^3)) \subset \mathbb{P}^9$);

(c) 3-minors of $3 \times (e + 2)$ sufficiently generic matrices (e.g. $Sec(RNS)$).

► Main question at this point is to give geometric classification/or characterization of ACM varieties with 3-linear resolution?

[Examples of varieties having ACM 3-linear resolution]

(a) Hypercubic ($e = 1$);

(b) 3-minors of 4×4 generic symmetric matrix (i.e. $\mathrm{Sec}(v_2(\mathbb{P}^3)) \subset \mathbb{P}^9$);

(c) 3-minors of $3 \times (e+2)$ sufficiently generic matrices (e.g. $Sec(RNS)$).

- Main question at this point is to give geometric classification/or characterization of ACM varieties with 3-linear resolution?

[Examples of varieties having ACM 3-linear resolution]

(a) Hypercubic ($e = 1$);

(b) 3-minors of 4×4 generic symmetric matrix (i.e. $\mathrm{Sec}(v_2(\mathbb{P}^3)) \subset \mathbb{P}^9$);

(c) 3-minors of $3 \times (e+2)$ sufficiently generic matrices (e.g. $Sec(RNS)$).

ND(k) subschemes

In general, we can define a ND(k) subscheme X in $\mathbb{P}^{n+e}$ whose Betti table is the following: (the k-th strand is the first nonzero strand!)

	0	1	2	3	$\cdots$	i	$i+1$	$\cdots$	$\triangle$
0	1	–	–	–	$\cdots$	–	–	$\cdots$	–
1	–	–	–	–	$\cdots$	–	–	$\cdots$	–
$\vdots$	–	–	$\ldots$	–	$\ddots$	$\ldots$	$\vdots$	$\vdots$	$\ldots$
k	–	$\beta_{1,k}$	$\beta_{2,k}$	$\beta_{3,k}$	$\cdots$	$\beta_{i,k}$	$\beta_{i+1,k}$	$\cdots$	$\beta_{\triangle,k}$
$k+1$	–	$\beta_{1,k+1}$	$\beta_{2,k+1}$	$\beta_{3,k+1}$	$\cdots$	$\beta_{i,k+1}$	$\beta_{i+1,k+1}$	$\cdots$	$\beta_{\triangle,k+1}$
$\vdots$			$\ldots$	–	$\ddots$	$\ldots$	$\vdots$	$\ldots$	$\ldots$

Example (ND(k)-subschemes)

- Algebraic sets whose one component is a ND(k)-scheme;
- ($k+1$)-minors of ($k+1$) $\times$ ($e+k$) sufficiently generic matrices (e.g. $\mathrm{Sec}^k(RNS)$).

$\mathrm{ND}(k)$ subschemes

In general, we can define a $\mathrm{ND}(k)$ subscheme X in $\mathbb{P}^{n+e}$ whose Betti table is the following: (the k-th strand is the first nonzero strand!)

	0	1	2	3	$\cdots$	i	$i+1$	$\cdots$	$\triangle$
0	1	–	–	–	$\cdots$	–	–	$\cdots$	–
1	–	–	–	–	$\cdots$	–	–	$\cdots$	–
$\vdots$	–	–	$\cdots$	–	$\ddots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$
k	–	$\beta_{1,k}$	$\beta_{2,k}$	$\beta_{3,k}$	$\cdots$	$\beta_{i,k}$	$\beta_{i+1,k}$	$\cdots$	$\beta_{\triangle,k}$
$k+1$	–	$\beta_{1,k+1}$	$\beta_{2,k+1}$	$\beta_{3,k+1}$	$\cdots$	$\beta_{i,k+1}$	$\beta_{i+1,k+1}$	$\cdots$	$\beta_{\triangle,k+1}$
$\vdots$			$\cdots$	–	$\ddots$	$\cdots$	$\vdots$	$\cdots$	$\cdots$

Example ($\mathrm{ND}(k)$-subschemes)

- Algebraic sets whose one component is a $\mathrm{ND}(k)$-scheme;
- $(k+1)$-minors of $(k+1)\times(e+k)$ sufficiently generic matrices (e.g. $\mathrm{Sec}^k(RNS)$).

Similarly, we have a following theorem:

Theorem (Ahn, Han and K-)

Suppose that $X^n \subset \mathbb{P}^{n+e}$ is a $\mathrm{ND}(k)$ subscheme, defined over $K = \overline{K}$ of char $(K) = 0$. Then,

- $\binom{e+k}{k} \leq \deg(X)$ *and* $h^0(\mathcal{I}_X(k)) \leq \binom{e+k}{k+1}$.
- *In general,* $\beta_{p,k}(X) \leq \binom{p+k-1}{k}\binom{e+k}{p+k}$ *for* $p \geq 1$.
- *For the extremal cases, the following are equivalent:*
  - (a) $\deg(X) = \binom{e+k}{k}$;
  - (b) $h^0(\mathcal{I}_X(k+1)) = \binom{e+k}{k+1}$;
  - (c) *one of* $\beta_{p,k}(X)$ *attains "=" for* $1 \leq p \leq e$;
  - (d) I_X *has ACM* $(k+1)$*-linear resolution.*

▶ There is a filtration of categories of $\mathrm{ND}(k)$-schemes:

$$\cdots \mathrm{ND}(k) \text{ schemes } \subset \cdots \subset \mathrm{ND}(2) \text{ schemes } \subset \mathrm{ND}(1) \text{ schemes }.$$

Similarly, we have a following theorem:

Theorem (Ahn, Han and K-)

Suppose that $X^n \subset \mathbb{P}^{n+e}$ *is a* $\mathrm{ND}(k)$ *subscheme, defined over* $K = \overline{K}$ *of* $\mathrm{char}\,(K) = 0$. *Then,*

- $\binom{e+k}{k} \leq \deg(X)$ *and* $h^0(\mathcal{I}_X(k)) \leq \binom{e+k}{k+1}$.
- *In general,* $\beta_{p,k}(X) \leq \binom{p+k-1}{k}\binom{e+k}{p+k}$ *for* $p \geq 1$.
- *For the extremal cases, the following are equivalent:*
  - (a) $\deg(X) = \binom{e+k}{k}$;
  - (b) $h^0(\mathcal{I}_X(k+1)) = \binom{e+k}{k+1}$;
  - (c) *one of* $\beta_{p,k}(X)$ *attains "=" for* $1 \leq p \leq e$;
  - (d) I_X *has ACM* $(k+1)$*-linear resolution.*

▶ There is a filtration of categories of $\mathrm{ND}(k)$-schemes:

$$\cdots \mathrm{ND}(k) \text{ schemes } \subset \cdots \subset \mathrm{ND}(2) \text{ schemes } \subset \mathrm{ND}(1) \text{ schemes } .$$

Similarly, we have a following theorem:

Theorem (Ahn, Han and K-)

Suppose that $X^n \subset \mathbb{P}^{n+e}$ is a $\mathrm{ND}(k)$ subscheme, defined over $K = \overline{K}$ of char $(K) = 0$. Then,

- $\binom{e+k}{k} \leq \deg(X)$ *and* $h^0(\mathcal{I}_X(k)) \leq \binom{e+k}{k+1}$.
- *In general,* $\beta_{p,k}(X) \leq \binom{p+k-1}{k}\binom{e+k}{p+k}$ *for* $p \geq 1$.
- *For the extremal cases, the following are equivalent:*
  - (a) $\deg(X) = \binom{e+k}{k}$;
  - (b) $h^0(\mathcal{I}_X(k+1)) = \binom{e+k}{k+1}$;
  - (c) *one of* $\beta_{p,k}(X)$ *attains "=" for* $1 \leq p \leq e$;
  - (d) I_X *has ACM* $(k+1)$*-linear resolution.*

▶ There is a filtration of categories of $\mathrm{ND}(k)$-schemes:

$$\cdots \mathrm{ND}(k) \text{ schemes } \subset \cdots \subset \mathrm{ND}(2) \text{ schemes } \subset \mathrm{ND}(1) \text{ schemes} .$$

Classification

- In each category of $\mathrm{ND}(k)$ subschemes, the minimal degree is $\binom{e+k}{k}$ and the minimal regularity is $k+1$.
- Only ACM varieties with $(k+1)$-linear resolution have minimal degree and minimal regularity.
- Classification of such varieties is very far from being complete.

Classification

- In each category of $\mathrm{ND}(k)$ subschemes, the minimal degree is $\binom{e+k}{k}$ and the minimal regularity is $k+1$.
- Only ACM varieties with $(k+1)$-linear resolution have minimal degree and minimal regularity.
- Classification of such varieties is very far from being complete.

Classification

- In each category of $\mathrm{ND}(k)$ subschemes, the minimal degree is $\binom{e+k}{k}$ and the minimal regularity is $k+1$.
- Only ACM varieties with $(k+1)$-linear resolution have minimal degree and minimal regularity.
- Classification of such varieties is very far from being complete.