Understanding of the defining equations and syzygies via inner projections and generic initial ideals

Sijong Kwak (KAIST, Korea)

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\( X \subset \mathbb{P}(V), \ V \subset H^0(\mathcal{O}_X(1)) \): a nondegenerate, irreducible and reduced variety of \( \dim(X) = n \) and \( \text{codim}(X) = e \) defined over \( K = \overline{K} \) of char \((K) \geq 0\). If \( V = H^0(\mathcal{O}_X(1)) \), one says that \( X \) is a linearly normal embedding.

\( R/I_X \): the projective coordinate ring of \( X \) where \( R = K[x_0, x_1, \ldots, x_{n+e}] \) is a coordinate ring of \( \mathbb{P}(V) \) and \( I_X = \bigoplus_{m \geq 0} H^0(\mathcal{I}_X(m)) \) is the saturated ideal.

There is a basic exact sequence:

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0 \to R/I_X \to \bigoplus_{m \geq 0} H^0(\mathcal{O}_X(m)) \to \bigoplus_{m \geq 0} H^1(\mathcal{I}_X(m)) \to 0
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and \( X \) is called "projectively normal" if \( \bigoplus_{m \geq 0} H^1(\mathcal{I}_X(m)) = 0 \).
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Basic Goal: Understand the minimal free resolutions of $R/I_X$ and $R(X) = \bigoplus_{m \geq 0} H^0(O_X(m))$ and their associated Betti tables in terms of geometric invariants.

Defining equations and their relations (called "syzygies") of $X$ appear in the (unique) minimal free resolution of $R/I_X$:

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\cdots \to L_i \to L_{i-1} \to \cdots \to L_1 \to R \to R/I_X \to 0 \quad \text{where} \quad L_i = \bigoplus_j R(-i - j) \beta_{i,j}(X).
$$

Note that $\beta_{i,j}(X)$ is the rank of the degree $i+j$ part in $L_i$ and $\beta_{i,j}(X) = \dim_K \text{Tor}_i^R(R/I_X, K)_{i+j}$.

The simplest nontrivial example is a rational normal curve $\nu_d(\mathbb{P}^1) \hookrightarrow \mathbb{P}^d$. How to compute the minimal free resolution of $\nu_d(\mathbb{P}^1) \subset \mathbb{P}^d$ for $d = 3, 4$ by hand or by using Macaulay 2?
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where $L_0 = R \oplus R(-1)^t \bigoplus_{j \geq 2} R(-j)^{\beta_{0,j}}$.

Note that $t = \text{codim}(V, H^0(\mathcal{O}_X(1)))$ and if $X$ is not linearly normal, the basis elements in $H^0(\mathcal{O}_X(1)) \setminus V$ should be generators of $R(X)$ in degree 1.

In particular, the minimal free resolution of the section ring $R(X)$ also encodes some geometric information on the embedding $X \hookrightarrow \mathbb{P}(V)$.  

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For example, if the section ring \( \bigoplus_{m \geq 0} H^0(\mathcal{O}_X(m)) \) of a smooth variety \( X \) has the following minimal free resolution:

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\cdots \to R(-2)^{\beta_{1,1}} \to R \oplus R(-1)^{t} \to \bigoplus_{m \geq 0} H^0(\mathcal{O}_X(m)) \to 0,
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i.e. as simple as possible up to the first syzygies, then we have the following properties (due to E. Park-K, 2005):

- \( X \) is \( k \)-normal for all \( k \geq t + 1 \);
- \( X \) is cut out by equations of degree at most \( t + 2 \);
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Note that if $t = 0$, then $X$ is projectively normal, and so it is trivial.
Let $X \subset \mathbb{P}^{n+e}$ be a variety of $\dim(X) = n$ and $\deg(X) = d$. We have the Betti table of $R/I_X$ associated to the minimal free resolution.

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
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<th>3</th>
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<th>$i-1$</th>
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$\triangle = $ the projective dimension of $R/I_X$, $\triangle \geq e$.

$\Box = \text{reg}(R/I_X) = \text{reg}(X) - 1$, where

$\text{reg}(X) := \min \{ \alpha \mid H^i(\mathbb{P}, I_X(\alpha - i)) = 0 \}$.

(Eisenbud-Goto Conjecture) $\Box \leq \deg(X) - \text{codim}(X) = d - e$. 
Let $X \subset \mathbb{P}^{n+e}$ be a variety of $\dim(X) = n$ and $\deg(X) = d$. We have the Betti table of $R/I_X$ associated to the minimal free resolution.

\[
\begin{array}{cccccccccc}
0 & 1 & 2 & 3 & \cdots & i-1 & i & i+1 & \cdots & \triangle \\
0 & 1 & - & - & - & \cdots & - & - & - & - \\
1 & - & \beta_{1,1} & \beta_{2,1} & \beta_{3,1} & \cdots & \beta_{i-1,1} & \beta_{i,1} & \beta_{i+1,1} & \cdots & \beta_{\triangle,1} \\
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\vdots & - & - & \cdots & - & \cdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
j & - & \beta_{1,j} & \beta_{2,j} & \beta_{3,j} & \cdots & \beta_{i-1,j} & \beta_{i,j} & \beta_{i+1,j} & \cdots & \beta_{\triangle,j} \\
\vdots & \vdots & \cdots & - & \cdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
\square & - & \beta_{1,\square} & \beta_{2,\square} & \beta_{3,\square} & \cdots & \beta_{i-1,\square} & \beta_{i,\square} & \beta_{i+1,\square} & \cdots & \beta_{\triangle,\square}
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(Eisenbud-Goto Conjecture) $\square \leq \deg(X) - \text{codim}(X) = d - e$. 

Sijong Kwak (KAIST, Korea) Understanding of the defining equations and $\square$ November 21, 2015 6 / 47
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1 & - & \beta_{1,1} & \beta_{2,1} & \beta_{3,1} & \cdots & \beta_{i-1,1} & \beta_{i,1} & \beta_{i+1,1} & \cdots & \beta_{\triangle,1} \\
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By the symmetry of Tor, the graded Betti numbers are also defined via the Koszul exact sequence of the base field $K$:

$V = K\langle x_0, \ldots, x_{n+e} \rangle$ be the $K$-vector space in $K[x_0, \ldots, x_{n+e}]$.
Then, $\text{Tor}_i^R(R/I_X, K)_{i+j}$ is the homology of the Koszul complex:

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Note that the Koszul complex is exact if $i > n + e + 1$ or $j >> 0$.

- $\beta_{1,1}(X)$: the number of quadrics $Q_i \in I_X$;
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One can say that $X$ satisfies property $N_{2,p}$ (or $N_p$) if $\beta_{i,j}(X) = 0$ for $1 \leq i \leq p$, $j \geq 2$.

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We are interested in the Betti numbers in the first linear strand.

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The simplest examples of curves

- **A rational normal curve $C \subset \mathbb{P}^{1+e}:**
  
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- **An elliptic normal curve $C \subset \mathbb{P}^{1+e}:**$ Similarly, we have 
  
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Castelnuovo’s simple proof.

\( \Gamma = X \cap \mathbb{P}^e \) is a set of \( d \)-points in general position for general \( \mathbb{P}^e \).
Since \( d \geq e + 1 \), take a subset \( \Gamma' = \{ p_1, p_1, \ldots, p_{e+1} \} \subset \Gamma \subset \mathbb{P}^e \).
\( h^0(\mathcal{I}_X(2)) \leq h^0(\mathcal{I}_\Gamma(2)) \leq h^0(\mathcal{I}_{\Gamma'}(2)) = \binom{e+2}{2} - (e + 1) = \binom{e+1}{2} \).

□

Further results in this method.

- (Fano, 1894) Unless \( X \) is VMD, \( h^0(\mathcal{I}_X(2)) \leq \binom{e+1}{2} - 1 \) and " = " holds iff \( X \) is a del Pezzo variety (i.e. arithmetically Cohen-Macaulay and \( \deg(X) = e + 2 \)).
- Note that the curve sections of a VMD and a del Pezzo variety are the rational normal curve, smooth elliptic normal curve and a rational nodal curve.
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Our observation using inner projections.

Consider \( \pi_q : X \to X_q \subset \mathbb{P}^{n+e-1} \) from a smooth point \( q \in X \):

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\text{Bl}_q(X) \cong \tilde{X} \\
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Lemma

\( X^n \subset \mathbb{P}^{n+e} \): irreducible and reduced (not necessarily smooth). Then, we have the following:

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For the second inequality, we assume a smooth point $q = (1, 0, \cdots, 0) \in X$.

A quadratic form vanishing on $X$ has no $x_0^2$-term and can be written by the (Gauss) elimination as follows:

$$x_0 L_1 + Q_1, \cdots, x_0 L_t + Q_t, \tilde{Q}_1, \cdots, \tilde{Q}_\mu$$

where $L_i$ is a linear form and $Q_i, \tilde{Q}_j$ are quadrics in $k[x_1, x_2, \cdots, x_{n+e}]$.

Note that $t \leq e$ and $\mu = h^0(\mathcal{I}_X(q)(2))$. Therefore, by successive inner projections from smooth points up to a hypersurface $Z$ in $\mathbb{P}^{n+1}$,

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More general inequality  (Han-K, 2015)

- $\beta_{i,1}(X) \leq \beta_{i,1}(X_q) + \beta_{i-1,1}(X_q) + \binom{e}{i}, \quad i \geq 1$.
- The equality holds for $i \leq p$ if $X$ satisfies property $N_{2,p}$.

Using the above inequality, we have the following:

**Theorem** $X \subset \mathbb{P}^{n+e}$: irreducible and reduced (not necessarily smooth).

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\beta_{p,1}(X) \leq p\left(\binom{e+1}{p+1}\right), \quad p \geq 1
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and the following are equivalent:

(a) $X$ is a variety of minimal degree;
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We also characterize Fano varieties as follows:

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Elimination mapping cone sequence and the partial elimination ideal theory due to M. Green are very useful to prove the above fundamental inequality on the Betti numbers $\beta_{i,1}(X)$ and $\beta_{i,1}(X_q)$.

**Elimination mapping cone sequence**

Let $S = k[x_1, \ldots, x_{n+e}] \subset R = k[x_0, x_1 \ldots, x_{n+e}]$

Let $M$ be a graded $R$-module (so, $M$ is also a graded $S$-module).

Then, we have a natural long exact sequence:

$$
\text{Tor}_i^R(M)_{i+j} \to \text{Tor}_{i-1}^S(M)_{i-1+j} \to \text{Tor}_{i-1}^S(M)_{i-1+j+1} \to \text{Tor}_{i-1}^R(M)_{i-1+j+1}
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whose connecting homomorphism is induced by the multiplication map $\times x_0 : M(-1) \to M$.

- This long exact sequence is useful to study the syzygies of projections.
- We can prove that if $X$ satisfies property $N_p$, then there is no $p + 2$-secant $p$-plane to $X$. (A proof is on the blackboard!)
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$$\text{Tor}_i^R(M)_{i+j} \to \text{Tor}_{i-1}^S(M)_{i-1+j} \to \text{Tor}_{i-1}^S(M)_{i-1+j+1} \to \text{Tor}_i^R(M)_{i-1+j+1}$$

whose connecting homomorphism is induced by the multiplication map $\times x_0 : M(-1) \to M$.

This long exact sequence is useful to study the syzygies of projections.

We can prove that if $X$ satisfies property $N_p$, then there is no $p+2$-secant $p$-plane to $X$. (A proof is on the blackboard!)
Elimination mapping cone sequence and the partial elimination ideal theory due to M. Green are very useful to prove the above fundamental inequality on the Betti numbers $\beta_{i,1}(X)$ and $\beta_{i,1}(X_q)$.

**Elimination mapping cone sequence**

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Varieties of minimal degree

$X^n \subset \mathbb{P}^{n+e}$: a variety (not necessarily smooth) of degree $d$.

Note that $d \geq e + 1$. $X$ is called "minimal degree variety" if $d = e + 1$.

The simplest Betti table with $\beta_{i,1} = i \cdot \binom{e+1}{i+1}$:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>$\cdots$</th>
<th>$i-1$</th>
<th>$i$</th>
<th>$i+1$</th>
<th>$\cdots$</th>
<th>$e$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$\cdots$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$\cdots$</td>
<td>$-$</td>
</tr>
<tr>
<td>1</td>
<td>$-$</td>
<td>$\beta_{1,1}$</td>
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Table 1  minimal degree varieties

A VMD has a rational normal curve section and they have the same Betti table.
**Varieties of minimal degree**

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<tbody>
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A VMD has a rational normal curve section and they have the same Betti table.
On the other hand, P. del Pezzo (1886) and E. Bertini (1907) classified all varieties of minimal degree:

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  (b) a Veronese surface $\nu_2(\mathbb{P}^2)$ in $\mathbb{P}^5$;
  
  (c) a rational normal scroll, i.e. $\mathbb{P}(\mathcal{E}) \hookrightarrow \mathbb{P}^{\Sigma a_i+d}$, where
  
  $\mathcal{E} \cong \bigoplus_{i=0}^d \mathcal{O}_{\mathbb{P}^1}(a_i)$, $a_i \geq 1$.

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\( X \) is called a **del Pezzo variety** if \( d = e + 2 \) and \( \text{depth}(X) = n + 1 \).

The (next-to-simplest) Betti table of a del Pezzo variety:

\[
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & \cdots & i & \cdots & e-1 & e \\
0 & 1 & - & - & - & \cdots & - & \cdots & - & - \\
1 & - & \beta_{1,1} & \beta_{2,1} & \beta_{3,1} & \cdots & \beta_{i,1} & \cdots & \beta_{e-1,1} & - \\
2 & - & - & - & - & \cdots & - & - & - & \beta_{e,2} = 1 \\
\end{array}
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Table 2  del Pezzo variety with \( \beta_{i,1}(X) = i\binom{e+1}{i+1} - \binom{e}{i-1} \).

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<th>$i$</th>
<th>...</th>
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<td>...</td>
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<td>-</td>
</tr>
<tr>
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T. Fujita classified smooth del Pezzo varieties into 8 types.

T. Fujita also showed that (singular) normal del Pezzo cases are divisors of some specific type in rational normal scrolls.

(M. Brodmann-P. Schenzel) Every non-normal del Pezzo variety $X$ comes from outer projection of a minimal degree variety $\widetilde{X}$ from a point $q$ in $\text{Sec}(\widetilde{X}) \setminus \widetilde{X}$ satisfying $\dim \Sigma_q(\widetilde{X}) = \dim \widetilde{X} - 1$ where the secant locus

$$\Sigma_q(\widetilde{X}) := \{ x \in \widetilde{X} \mid \pi_q^{-1}(\pi_q(x)) \text{ has length at least 2} \}.$$ 

Thus, a non-normal del Pezzo variety has a rational nodal curve section.
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(i) Outer projections of a rational normal curve $C \subset \mathbb{P}^a$ $(a > 2)$ from $q \in \text{Sec}(C)$,
(ii) Outer projections of the Veronese surface $v_2(\mathbb{P}^2) \subset \mathbb{P}^5$ from $q \in \text{Sec}(v_2(\mathbb{P}^2))$,
(iii) Outer projections of a smooth cubic surface scroll $S(1, 2)$ in $\mathbb{P}^4$,
(iv) Outer projections of a smooth rational normal scroll $S(1, b)$ in $\mathbb{P}^{b+2}$ $(b > 2)$ from $q \in \text{Join}(S(1), S(1, b))$,
(v) Outer projections of a smooth quartic surface scroll $S(2, 2)$ in $\mathbb{P}^5$ from $q \in \mathbb{P}^2 \times \mathbb{P}^1$,
(vi) Outer projections of a smooth surface scroll $S(2, b)$ in $\mathbb{P}^{b+3}$ $(b > 2)$ from $q \in \langle S(2) \rangle$,
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Asymptotic behavior of $\beta_{p,1}$ for smooth curves

Let $C$ be a smooth curve of genus $g$ and the gonality $g_0$. Suppose $\text{deg}(\mathcal{L}) \geq 4g - 3$ and $r = h^0(C, \mathcal{L}) - 1$.

1. $\beta_{p,1}(C) \neq 0 \iff 1 \leq p \leq r - g_0$ (Ein-Lazarsfeld);
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<table>
<thead>
<tr>
<th></th>
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<th>1</th>
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<th>$r - g - 1$</th>
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<td>$\beta_{r-g,1}$</td>
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</table>

We can also get the upper bound of $\beta_{p,1}$ by using inner projection method as follows:

$$\beta_{p,1}(C) \leq p \left( \frac{e + 1}{p + 1} \right) + \left( \frac{e + 1}{p + 1} \right) - \left( \frac{e + 2 - g_0}{p + 1} \right) - (g_0 - 1) \left( \frac{e + 1}{p} \right).$$
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\[
\begin{array}{cccccccc}
0 & 1 & \cdots & r - g - 1 & r - g & \cdots & r - g_0 & \cdots & r - 1 \\
0 & 1 & - & \cdots & - & \cdots & - & \cdots & - \\
1 & - & \beta_{1,1} & \cdots & \beta_{r - g - 1,1} & \beta_{r - g,1} & \cdots & \beta_{r - g_0,1} & - & - \\
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| 0 | 1 | $\cdots$ | $r - g - 1$ | $r - g$ | $\cdots$ | $r - g_0$ | $\cdots$ | $r - 1$
|---|---|---|---|---|---|---|---|---
| 0 | 1 | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$
| 1 | $\cdots$ | $\beta_{1,1}$ | $\cdots$ | $\beta_{r-g-1,1}$ | $\beta_{r-g,1}$ | $\cdots$ | $\beta_{r-g_0,1}$ | $\cdots$ | $\cdots$
| 2 | $\cdots$ | $\beta_{1,1}$ | $\cdots$ | $\cdots$ | $\beta_{r-g,2}$ | $\cdots$ | $\cdots$ | $\beta_{r-1,2}$

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<td>$r - g - 1$</td>
<td>$r - g$</td>
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<td>$\beta_{r-1,2}$</td>
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</table>
Asymptotic behavior of $\beta_{p,1}$ for smooth curves

Let $C$ be a smooth curve of genus $g$ and the gonality $g_0$. Suppose $\deg(L) \geq 4g - 3$ and $r = h^0(C, L) - 1$.

(a) $\beta_{p,1}(C) \neq 0 \iff 1 \leq p \leq r - g_0$ (Ein-Lazarsfeld);
(b) $\beta_{p,2}(C) \neq 0 \iff r - g \leq p \leq r - 1$ (Green and Schreyer).

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<tr>
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<th>$r - g - 1$</th>
<th>$r - g$</th>
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We can also get the upper bound of $\beta_{p,1}$ by using inner projection method as follows:

$$\beta_{p,1}(C) \leq p \left( \frac{e + 1}{p + 1} \right) + \left( \frac{e + 1}{p + 1} \right) - \left( \frac{e + 2 - g_0}{p + 1} \right) - (g_0 - 1) \left( \frac{e + 1}{p} \right).$$
The graded Betti table of $R/I_X$

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In the first linear strand, we summarize the following facts:

- $\beta_{p,1} = 0$ for $p > e$;
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The structure of the Betti table II (Regularity)

Let $X^n \subset \mathbb{P}^{n+e}$ be a non-degenerate projective variety of dim $n$, codim $e$, and degree $d$, and $H$ be a general hyperplane section.

**Definition**

1. $X$ is called $m$-regular if the following two conditions hold:
   1. $H^0(\mathcal{O}_{\mathbb{P}^{n+e}}(m-1)) \twoheadrightarrow H^0(\mathcal{O}_X(m-1))$ is surjective, i.e. $X$ is $(m-1)$-normal;
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2. $\text{reg}(X)$ (Castelnuovo-Mumford regularity of $X$) is the smallest number $m$ such that $X$ is $m$-regular;

**Geometric Regularity Bound**

$$\text{reg}(X) \leq d - e + 1$$ (Eisenbud-Goto conjecture).
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Proposition (Birational double point formula)
Let \( \varphi : V^n \rightarrow M^{n+1} \) be a morphism of smooth projective varieties such that \( \varphi : V \rightarrow W := \varphi(V) \subset M \) is birational. Then,

\[
\varphi^*(K_M + W) - K_V \sim D - E,
\]

where \( D \) and \( E \) are effective divisors on \( V \) such that \( E \) is \( \varphi \)-exceptional. Moreover, if \( \varphi \) is isomorphic at \( x \in V \), then \( x \notin \text{Supp}(D - E) \).

Proof. see Lemma 10.2.8. in Positivity in Algebraic Geometry II.

▶ We apply this formula to a general projection of smooth varieties:

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\pi_\Lambda : X \rightarrow X_\Lambda \subset \mathbb{P}^{n+1}, \Lambda = \mathbb{P}^{e-2} \text{ and } \Lambda \cap X = \emptyset.
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Let $X$ be smooth and $\mathcal{L}$ is very ample. We have the following:

$$H^i(X, \mathcal{L} \otimes (d-e-i)) = H^i(\mathcal{O}_X(d-e-i)) = 0, \ i \geq 1$$

where $d = \text{deg}(X)$ and $e = \text{codim}(X)$ in the embedding of $X \subset \mathbb{P}(H^0(\mathcal{L}))$.

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Duality Theorem for syzygies
Consider the following diagram

\[ \begin{align*}
D_\Lambda & \subset X & \xrightarrow{\pi_\Lambda} & \mathbb{P}^{n+e} \\
\downarrow \pi_\Lambda & \downarrow \pi_\Lambda & \uparrow & \vdots \\
Z_\Lambda & \subset X_\Lambda & \xrightarrow{} & \mathbb{P}^{n+1}.
\end{align*} \]

Since \( \pi_\Lambda \) is finite, so \( \pi_\Lambda \)-exceptional divisor \( E = \emptyset \) and the non-isomorphic locus \( D_\Lambda \) (as a divisor) is linearly equivalent to

\[ B_2 := (d - n - 2)H - K_X, \]

which is called a double point divisor arising from \( \pi_\Lambda : X \to X_\Lambda \subset \mathbb{P}^{n+1} \).

Let \( V \subset H^0(O_X(B_2)) \) be a subspace spanned by geometric sections (i.e. for \( s \in V \), \( \text{div}(s) = D_\Lambda \) coming from an actual outer projection).
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$V$ is a basepoint-free subsystem in $H^0(\mathcal{O}_X(B_2))$. In particular,

$$H^i(\mathcal{O}_X(d - i - 1)) = 0, \quad i \geq 1.$$ 

**Proof.** For $x \in X$, take a general linear space $\Lambda_x$ such that $\Lambda_x \cap (T_x(X)^n \cup \text{Cone}(x, X)^{n+1}) = \emptyset$. Then $\pi_{\Lambda_x} : X \rightarrow X_{\Lambda_x} \subset \mathbb{P}^{n+1}$ is isomorphic near $x$ and $x \notin D_{\Lambda_x}$ and $B_2 := (d - n - 2)H - K_X$ is basepoint-free. Since $(d - 1 - i)H = K_X + (n + 1 - i)H + B_2$, by Kodaira vanishing theorem $H^i(\mathcal{O}_X(d - i - 1)) = 0, \quad i \geq 1$. 

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Proposition (Mumford)

$V$ is a basepoint-free subsystem in $H^0(\mathcal{O}_X(B_2))$. In particular,

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B. Ilic (1995) proved that $B_2$ is big unless $X$ is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^{n-1}$. By Kawamata-Viehweg vanishing, we have

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**Double point divisors from inner projections**

Let $x_1, \ldots, x_{e-1} \in X$ be general points, and let $\Lambda := \langle x_1, \ldots, x_{e-1} \rangle$. Consider the inner projection at $\Lambda$ and the blow-up at $x_1, \ldots, x_{e-1}$.

Note that $\deg(\overline{X}_\Lambda) = \deg(X) - (e - 1) = d - (e - 1)$. 
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![Diagram of inner projection and blow-up](attachment:inner_projection_diagram.png)

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Assume that $X$ is neither a scroll over a curve nor a second Veronese surface. Then $\tilde{\pi} : \tilde{X} \rightarrow \tilde{X}_\Lambda$ has no exceptional divisor (A. Noma, 2013). So, by the birational double point formula, we obtain an effective divisor $D(\tilde{\pi})$ which is the non-isomorphic locus of $\tilde{\pi}$. Then,

$$D(\pi) := \sigma(D(\tilde{\pi})|_{\tilde{X} \setminus E_1 \cup \cdots \cup E_{e-1}}$$

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If $X$ is neither a scroll over a curve, a second Veronese surface, nor a Roth variety, then, $C(X)$ is finite and we have the following:

**Proposition (Noma)**

$B_{2,\text{inn}}$ is semiample, i.e. some power of $B_{2,\text{inn}}$ is basepoint-free.

**Proof:** For any point $x \in X \setminus C(X)$, by varying centers, we can take an inner projection $\pi : X \to X_\Lambda \subset \mathbb{P}^{n+1}$ isomorphic at $x$. □

Note that $\text{b.p.f.} \Rightarrow \text{semiample} \Rightarrow \text{nef.}$

**Corollary**

$\text{reg}_H(O_X) \leq d - e$ if $X$ is neither a scroll over a curve, a second Veronese surface, nor a Roth variety.

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Understanding of the defining equations and syzygies via inner projections and generic initial ideals

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Proposition

We have the following $O_X$-regularity bound for smooth varieties:

- $X = v_2(P^2) \subset P^4$ or in $P^5 \Rightarrow \text{reg}(O_X) = 1$;
- $X$ : Roth variety $\Rightarrow \text{reg}(O_X) \leq d - e$;
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Therefore, the second part of Eisenbud-Goto conjecture (i.e. $\text{reg}(O_X) \leq d - e$) is proved for smooth varieties. Thus,

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Generalization to $\text{ND}(1)$-subscheme

Our results can be generalized to $\text{ND}(1)$-subschemes using generic initial ideals in graded reverse lexicographic order.

Definition

A closed subscheme $X^n \subset \mathbb{P}^{n+e}$ is called $\text{ND}(1)$-subscheme if $X \cap \Lambda$ is nondegenerate for a general $\Lambda$ of dimension $e \leq \dim \Lambda \leq n + e$.

A $\text{ND}(1)$-subscheme is not necessarily to be irreducible, reduced or equi-dimensional in general.

Example ($\text{ND}(1)$-subschemes)

- Nondegenerate varieties;
- Connected in codimension 1 algebraic sets, i.e. $X$ is equidimensional and each components are ordered such that $X_i \cap X_{i+1}$ is of codim 1 in $X$;
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Non–ND(1) subschemes

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<table>
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<th>0</th>
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<th>2</th>
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This is a counter-example for all the previous results.

- Two planes meeting at one point in $\mathbb{P}^4$ is also non-ND(1).
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\begin{array}{cccc}
0 & 1 & 2 & 3 \\
0 & 1 & - & - & - \\
1 & - & 4 & 4 & 1 \\
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\[
\begin{array}{cccc}
0 & 1 & 2 & 3 \\
0 & 1 & - & - \\
1 & - & 4 & 4 & 1 \\
\end{array}
\]

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$X^n \subset \mathbb{P}^{n+e}$: a ND(1) subscheme, defined over $K = \overline{K}$ of char $(K) = 0$. Then, we have the following upper bound:

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Ideas of a proof in case $\text{char}(K) = 0$

**Generic initial ideal**

For $g = (g_{ij}) \in GL_{n+1}(K)$, consider $g(I) = \{g \cdot f | f \in I\}$ where $g \cdot f = f(gx_0, gx_1, \cdots, gx_{n+e})$ and $gx_i = \sum_{0 \leq k \leq n+e} g_{ik}x_k$.

Then, due to Galligo and Bayer-Stillman, $\text{in}_\tau (g(I))$ is constant for a general change $g$. We will call this the *generic initial ideal of $I$ w.r.t $\tau$*.

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Then, the Hilbert functions of $R/l_X$ and $R/\text{Gin}(l_X)$ are the same.

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where

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Our problems on syzygies on cubic generators

Let $X^n \subset \mathbb{P}^{n+e}$ be any variety over $K$. Suppose that $(I_X)_2 = 0$. Then,

(a) What can we say about upper bounds for $\beta_{p,2}(X)$ for any $p \geq 2$ and $K_{p,2}$ Theorem (i.e., $\beta_{p,2}(X) = 0$ for $p > e$ under some conditions.

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A closed subscheme $X^n \subset \mathbb{P}^{n+e}$ is called **ND(2)-subscheme** if for a general $\Lambda$ of dimension $e \leq \dim \Lambda \leq n + e$, $X \cap \Lambda$ is not contained in a quadric.
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<td>$\beta_{1,2}$</td>
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<td>$j$</td>
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Upper bound of $\beta_{p,2}(X)$ and $K_{p,2}$ Theorem

Theorem (Ahn, Han and K-, preprint)

Suppose that $X^n \subset \mathbb{P}^{n+e}$ is a ND(2) subscheme, defined over $K = \overline{K}$ of char $(K) = 0$. Then,

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- In general, $\beta_{p,2}(X) \leq \left(\frac{p+1}{2}\right)\left(\frac{e+2}{p+2}\right)$ for $p \geq 1$.

- For the extremal cases, the following are equivalent:
  - (a) $\deg(X) = \left(\frac{e+2}{2}\right)$;
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  - (c) one of $\beta_{p,2}(X)$ attains “=” for $1 \leq p \leq e$;
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This also gives a natural $K_{p,2}$ theorem generalizing $K_{p,1}$-theorem because $\beta_{p,2}(X) = 0$ for $p > e$. 
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This is a minimal degree variety in a category of ND(1)-varieties.

\[
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & \cdots & i & \cdots & e-1 & e \\
0 & 1 & - & - & - & \cdots & - & \cdots & - & - \\
1 & - & \beta_{1,1} & \beta_{2,1} & \beta_{3,1} & \cdots & \beta_{i,1} & \cdots & \beta_{e-1,1} & \beta_{e,1} \\
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ACM varieties with 2-linear or 3 linear resolutions

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<th>e - 1</th>
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<tr>
<td>0</td>
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Examples of varieties having ACM 3-linear resolution

(a) Hypercubic ($e = 1$);
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In general, we can define a $\text{ND}(k)$ subscheme $X$ in $\mathbb{P}^{n+e}$ whose Betti table is the following: (the $k$-th strand is the first nonzero strand!)

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1 & - & - & - & - & \cdots & - & - & \cdots & - \\
\vdots & - & - & \cdots & - & \vdots & \cdots & \vdots & \vdots & \vdots \\
k & - & \beta_{1,k} & \beta_{2,k} & \beta_{3,k} & \cdots & \beta_{i,k} & \beta_{i+1,k} & \cdots & \beta_{\triangle,k} \\
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\vdots & \cdots & - & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
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Similarly, we have a following theorem:

**Theorem (Ahn, Han and K-)**

Suppose that $X^n \subset \mathbb{P}^{n+e}$ is a $\text{ND}(k)$ subscheme, defined over $K = \overline{K}$ of char $(K) = 0$. Then,

- $\binom{e+k}{k} \leq \text{deg}(X)$ and $h^0(\mathcal{I}_X(k)) \leq \binom{e+k}{k+1}$.
- In general, $\beta_{p,k}(X) \leq \binom{p+k-1}{k} \binom{e+k}{p+k}$ for $p \geq 1$.
- For the extremal cases, the following are equivalent:
  1. $\text{deg}(X) = \binom{e+k}{k}$;
  2. $h^0(\mathcal{I}_X(k+1)) = \binom{e+k}{k+1}$;
  3. one of $\beta_{p,k}(X)$ attains “=” for $1 \leq p \leq e$;
  4. $I_X$ has ACM $(k+1)$-linear resolution.

▶ There is a filtration of categories of $\text{ND}(k)$-schemes:

$$
\cdots \text{ND}(k) \text{ schemes} \subset \cdots \subset \text{ND}(2) \text{ schemes} \subset \text{ND}(1) \text{ schemes}.
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In each category of $\mathcal{N}\mathcal{D}(k)$ subschemes, the minimal degree is $\binom{e+k}{k}$ and the minimal regularity is $k + 1$.

Only ACM varieties with $(k + 1)$-linear resolution have minimal degree and minimal regularity.

Classification of such varieties is very far from being complete.
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